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Singular Abreu equations and linearized Monge–Ampère equations with drifts

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Abstract. We study the solvability of singular Abreu equations which arise in the approximation of convex functionals subject to a convexity constraint. Previous works established the solvability of their second boundary value problems either in two dimensions, or in higher dimensions under either a smallness condition or a radial symmetry condition. Here, we solve the higher-dimensional case by transforming singular Abreu equations into linearized Monge–Ampère equations with drifts. We establish global Hölder estimates for linearized Monge–Ampère equations with drifts under suitable hypotheses, and then apply them to prove the regularity and solvability of the second boundary value problem for singular Abreu equations in higher dimensions. Many cases with general right-hand side are also discussed.

Keywords: singular Abreu equation, linearized Monge–Ampère equation with drift, second boundary value problem, Monge–Ampère equation, Legendre transform, pointwise Hölder estimate.

1. Introduction and statements of the main results

In this paper, we study the solvability of the second boundary value problem of the following fourth order Monge–Ampère type equation on a bounded, smooth, uniformly convex domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$:

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$$\begin{cases} \sum_{i,j=1}^{n} U^{ij} D_{ij} w = -\gamma \operatorname{div}(|Du|^{q-2} Du) + \mathbf{b} \cdot Du + c(x,u) \\ \qquad\qquad =: f(x,u,Du,D^2u) & \text{in } \Omega, \\ w = (\det D^2 u)^{-1} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\gamma \geq 0$, $q > 1$, $U = (U^{ij})_{1\leq i,j\leq n}$ is the cofactor matrix of the Hessian matrix

$$D^2 u = (D_{ij} u)_{1\leq i,j\leq n} \equiv \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{1\leq i,j\leq n}$$

of an unknown uniformly convex function $u \in C^2(\overline{\Omega})$, $\varphi \in C^{3,1}(\overline{\Omega})$, $\psi \in C^{1,1}(\overline{\Omega})$, $\mathbf{b} : \overline{\Omega} \to \mathbb{R}^n$ is a vector field on $\overline{\Omega}$, and $c(x,z)$ is a function on $\overline{\Omega} \times \mathbb{R}$. When the right-hand side f depends only on the independent variable, that is, $f = f(x)$, (1.1) is the *Abreu equation* arising from the problem of finding extremal metrics on toric manifolds in Kähler geometry [1], and it is equivalent to

$$\sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = f(x),$$

where (u^{ij}) is the inverse matrix of $D^2 u$. The general form in (1.1) was introduced by the second author in [22–24] in the study of convex functionals with a convexity constraint related to the Rochet–Choné model [31] for the monopolist's problem in economics, whose Lagrangian depends on the gradient variable; see also Carlier–Radice [4] for the case where the Lagrangian does not depend on the gradient variable.

More specifically, in the calculus of variations with a convexity constraint, one considers minimizers of convex functionals

$$\int_{\Omega} F_0(x, u(x), Du(x))\, dx$$

among certain classes of convex competitors, where $F_0(x, z, \mathbf{p})$ is a function on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. One example is the Rochet–Choné model with q-power ($q > 1$) cost,

$$F_{q,\gamma}(x, z, \mathbf{p}) = (|\mathbf{p}|^q/q - x \cdot \mathbf{p} + z)\gamma(x),$$

where γ is a nonnegative Lipschitz function called the relative frequency of agents in the population.

Since it is in general difficult to handle the convexity constraint, especially in numerical computations [2, 30], instead of analyzing these functionals directly, one might consider analyzing their perturbed versions by adding the penalizations $-\varepsilon \int_{\Omega} \log\det D^2 u\, dx$ which are convex functionals in the class of C^2, strictly convex functions. The heuristic idea is that the logarithm of the Hessian determinant should act as a good barrier for the convexity constraint. This was verified numerically in [2] at a discretized level. Note that critical points, with respect to compactly supported variations, of the convex functional

$$\int_\Omega F_0(x, u(x), Du(x))\, dx - \varepsilon \int_\Omega \log\det D^2u\, dx$$

satisfy the Abreu type equation

$$\varepsilon U^{ij} D_{ij}[(\det D^2u)^{-1}] = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial F_0}{\partial p_i}(x, u, Du)\right) + \frac{\partial F_0}{\partial z}(x, u, Du).$$

Here we denote $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n$. In particular, for the Rochet–Choné model with q-power ($q > 1$) cost and unit frequency $\gamma \equiv 1$, that is, $F_0 = F_{q,1}$, the above right-hand side is

$$-\operatorname{div}(|Du|^{q-2}Du) + n + 1,$$

which belongs to the class of right-hand sides considered in (1.1). When $F_0(x, z, \mathbf{p}) = F(\mathbf{p}) + \hat{F}(x, z)$ the above right-hand side becomes

$$-\operatorname{div}(DF(Du)) + \frac{\partial \hat{F}}{\partial z}(x, u).$$

When $\gamma > 0$, we call (1.1) a *singular Abreu equation* because its right-hand side depends on D^2u which can be just a matrix-valued measure for a merely convex function u.

Our focus in this paper will be on the case $\gamma > 0$. For simplicity, we will take $\gamma = 1$.

The Abreu type equations can be included in the class of fourth order Monge–Ampère type equations of the form

$$U^{ij} D_{ij}[g(\det D^2u)] = f \tag{1.2}$$

where $g : (0, \infty) \to (0, \infty)$ is an invertible function. In particular, when $g(t) = t^\theta$, one can take $\theta = -1$ and $\theta = -\frac{n+1}{n+2}$ to get the Abreu type equation and the *affine mean curvature* type equation [7], respectively. It is convenient to write (1.2) as a system of two equations for u and $w = g(\det D^2u)$. One is a Monge–Ampère equation for the convex function u in the form of

$$\det D^2u = g^{-1}(w) \tag{1.3}$$

and the other is the following linearized Monge–Ampère equation for w:

$$U^{ij} D_{ij} w = f. \tag{1.4}$$

The second order linear operator $\sum_{i,j=1}^n U^{ij} D_{ij}$ is the linearized Monge–Ampère operator associated with the convex function u because its coefficient matrix comes from linearizing the Monge–Ampère operator:

$$U = \frac{\partial \det D^2u}{\partial(D^2u)}.$$

When u is sufficiently smooth, such as $u \in W^{4,s}_{\mathrm{loc}}(\Omega)$ where $s > n$, the expression

$\sum_{i,j=1}^n U^{ij} D_{ij} w$ can be written as $\sum_{i,j=1}^n D_i(U^{ij} D_j w)$, since the cofactor matrix (U^{ij}) is divergence-free, that is,

$$\sum_{i=1}^n D_i U^{ij} = 0$$

for all j. The regularity and solvability of equation (1.2), under suitable boundary conditions, are closely related to the regularity theory of the linearized Monge–Ampère equation, initiated in the fundamental work of Caffarelli–Gutiérrez [3]. In the past two decades, there has been much progresses in the study of these equations and related geometric problems, including [5, 6, 9–12, 17, 18, 37–41], to name but a few.

According to the decomposition (1.3) and (1.4), a very natural boundary value problem for the class of fourth order equations (1.2) is the second boundary value problem where one prescribes the values of u and w on the boundary $\partial\Omega$ as in (1.1).

1.1. Previous results and difficulties

A summary of solvability results for (1.1), or more generally, the second boundary value problem for (1.2), for the case $f \equiv f(x)$ is as follows. For the second boundary value problem of the affine mean curvature equation, that is, (1.2) with $g(t) = t^{-\frac{n+1}{n+2}}$, Trudinger–Wang [38, 39] proved the existence of a unique $C^{4,\alpha}(\bar{\Omega})$ solution when $f \in C^{\alpha}(\bar{\Omega})$ with $f \leq 0$, and a unique $W^{4,p}(\Omega)$ solution when $f \in L^{\infty}(\Omega)$ with $f \leq 0$. The analogous result for the Abreu equation (1.1) was then obtained by the fourth author [41]. For the $W^{4,p}(\Omega)$ solution, the second author [17] solved (1.1) for $f \in L^p(\Omega)$ with $p > n$ and $f \leq 0$. The sign condition on f was removed by Chau-Weinkove [5] under the assumption that $f \in L^p(\Omega)$ with $p > n$ and $f^+ := \max\{f, 0\} \in L^q(\Omega)$ with $q > n + 2$ for the affine mean curvature equation. Finally, in [18], the second author showed that the $W^{4,p}(\Omega)$ solution exists under the weakest assumption $f \in L^p(\Omega)$ with $p > n$ for a broad class of equations like (1.2), including both the affine mean curvature equation and the Abreu equation. We will concentrate on the singular Abreu equation (1.1), and its solvability in $C^{4,\alpha}$ and $W^{4,s}$ $(s > n)$. We obtain solvability by establishing a priori higher order derivative estimates and then using degree theory. Essentially, establishing a priori estimates requires establishing Hessian determinant estimates for u and Hölder estimates for w.

For the singular Abreu equation, the dependence of the right-hand side on D^2u creates two new difficulties in applying the regularity theory of the linearized Monge–Ampère equation. The first difficulty lies in obtaining a priori lower and upper bounds for $\det D^2u$, which is a critical step in applying the regularity results for the linearized Monge–Ampère equation. The appearance of D^2u has very subtle effects on Hessian determinant estimates. The second author [22] obtained Hessian determinant estimates for $f = -\operatorname{div}(|Du|^{q-2}Du)$ in two dimensions with $q \geq 2$ by using a special algebraic structure of the equation. In a recent work of the second and fourth authors [28], Hessian determinant estimates for the case $1 < q < 2$ were established by using partial Legendre transform. The second difficulty, granted that the bounds $0 < \lambda \leq \det D^2u \leq \Lambda < \infty$ have been established, consists in obtaining Hölder estimates for w in the linearized Monge–Ampère equation (1.4), which has no lower order terms on the left-hand side. This requires a certain integrability condition for the right-hand side, as can be seen from the simple equation $\Delta w = f$. In previous works [3, 15, 16], classical regularity estimates for the linearized Monge–Ampère equation were obtained for L^n right-hand side. This integrability breaks down even in the case $f = -\Delta u$ (where $q = 2$, $\mathbf{b} = 0$ and $c = 0$), which is a priori at most $L^{1+\varepsilon}$ for some small constant $\varepsilon(\lambda, \Lambda, n) > 0$ (see [8, 13, 35]). With the Hölder estimates for the linearized Monge–Ampère equation with $L^{n/2+\varepsilon}$ right-hand side in [26], the second author [22] established the solvability of (1.1) for $f = -\operatorname{div}(|Du|^{q-2}Du)$ in two dimensions with $q \geq 2$. When $1 < q < 2$, $f = -\operatorname{div}(|Du|^{q-2}Du)$ becomes more singular in D^2u and hence it has lower integrability (if any). However, in two dimensions, the second and fourth authors [28] solved the second boundary value problem (1.1) for $f = -\operatorname{div}(|Du|^{q-2}Du) + c(x, u)$ for any $q > 1$ under suitable assumptions on c and the boundary data. The proof was based on the interior and global Hölder estimates for the linearized Monge–Ampère equation with the right-hand side being the divergence of a bounded vector field which were established in [19, 21]. The solvability of the singular Abreu equations (1.1) in higher dimensions, even in the simplest case $f = -\Delta u$, has been widely open. Only some partial results were obtained in [24] under either a smallness condition (such as replacing $f = -\Delta u$ by $f = -\delta\Delta u$ for a suitably small constant $\delta > 0$) or a radial symmetry condition.

1.2. Statements of the main results

The purpose of this paper is to solve the higher-dimensional case of (1.1). We will first consider the case where the right-hand side has no drift term $\mathbf{b} \cdot Du$. This case answers in the affirmative the question raised in [28, p. 6]. In fact, we can establish the solvability for singular Abreu equations that are slightly more general than (1.1) where $\operatorname{div}(|Du|^{q-2}Du)$ is replaced by $\operatorname{div}(DF(Du))$ for a suitable convex function F. Our first main theorem is as follows.

Theorem 1.1 (Solvability of the second boundary value problem for singular Abreu equations in higher dimensions). *Let $\Omega \subset \mathbb{R}^n$ be an open, smooth, bounded and uniformly convex domain. Let $r > n$. Let $F \in W^{2,r}_{\mathrm{loc}}(\mathbb{R}^n)$ be a convex function. Assume that $\varphi \in C^5(\overline{\Omega})$ and $\psi \in C^3(\overline{\Omega})$ with $\min_{\partial\Omega} \psi > 0$. Consider the following second boundary value problem for a uniformly convex function u:*

$$\begin{cases} \sum_{i,j=1}^{n} U^{ij} D_{ij} w = -\operatorname{div}(DF(Du)) + c(x, u) & \text{in } \Omega, \\ w = (\det D^2u)^{-1} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Here $(U^{ij}) = (\det D^2u)(D^2u)^{-1}$ and $c(x, z) \leq 0$.

(i) *Assume* $c \in C^{\alpha}(\overline{\Omega} \times \mathbb{R})$ *where* $\alpha \in (0, 1)$. *Then there exists a uniformly convex solution* $u \in W^{4,r}(\Omega)$ *to* (1.5) *with*

$$\|u\|_{W^{4,r}(\Omega)} \leq C$$

for some $C > 0$ *depending on* Ω, n, α, F, r, c, φ *and* ψ.

Moreover, if $F \in C^{2,\alpha_0}(\mathbb{R}^n)$ *where* $\alpha_0 \in (0, 1)$, *then there exists a uniformly convex solution* $u \in C^{4,\beta}(\overline{\Omega})$ *to* (1.5) *where* $\beta = \min\{\alpha, \alpha_0\}$ *with*

$$\|u\|_{C^{4,\beta}(\overline{\Omega})} \leq C$$

for some $C > 0$ *depending on* Ω, n, α, α_0, F, c, φ *and* ψ.

(ii) *Assume* $c(x, z) \equiv c(x) \in L^p(\Omega)$ *with* $p > n$ *where* $c(x) \leq 0$. *Then, for* $s = \min\{r, p\}$, *there exists a uniformly convex solution* $u \in W^{4,s}(\Omega)$ *to* (1.5) *with*

$$\|u\|_{W^{4,s}(\Omega)} \leq C$$

for some $C > 0$ *depending on* Ω, n, p, F, r, s, $\|c\|_{L^p(\Omega)}$, φ *and* ψ.

We will prove Theorem 1.1 in Section 4.

We also discuss the solvability and regularity estimates of (1.1) when the right-hand side has more general lower order terms and no sign restriction on c. We mainly focus on the most typical case that the right-hand side has a Laplace term:

$$\begin{cases} \sum_{i,j=1}^{n} U^{ij} D_{ij} w = -\Delta u + \mathbf{b} \cdot Du + c(x, u) & \text{in } \Omega, \\ w = (\det D^2 u)^{-1} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Here, $(U^{ij}) = (\det D^2 u)(D^2 u)^{-1}$. Our second main result is the following theorem.

Theorem 1.2 (Solvability of the second boundary value problem for singular Abreu equations with lower order terms in high dimensions). *Let* $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ *be an open, smooth, bounded and uniformly convex domain. Assume that* $\varphi \in C^5(\overline{\Omega})$ *and* $\psi \in C^3(\overline{\Omega})$ *with* $\min_{\partial\Omega} \psi > 0$. *Consider the second boundary value problem* (1.6) *with* $c(x, z) \equiv c(x)$.

(i) *If* $\mathbf{b} \in C^{\alpha}(\overline{\Omega}; \mathbb{R}^n)$ *and* $c \in C^{\alpha}(\overline{\Omega})$ *where* $\alpha \in (0, 1)$, *then there exists a uniformly convex solution* $u \in C^{4,\alpha}(\overline{\Omega})$ *to* (1.6) *with*

$$\|u\|_{C^{4,\alpha}(\overline{\Omega})} \leq C$$

for some $C > 0$ *depending on* Ω, n, α, $\|\mathbf{b}\|_{C^{\alpha}(\overline{\Omega})}$, $\|c\|_{C^{\alpha}(\overline{\Omega})}$, φ *and* ψ.

(ii) *If* $\mathbf{b} \in L^{\infty}(\Omega; \mathbb{R}^n)$ *and* $c \in L^p(\Omega)$ *with* $p > 2n$, *then there exists a uniformly convex solution* $u \in W^{4,p}(\Omega)$ *to* (1.6) *with*

$$\|u\|_{W^{4,p}(\Omega)} \leq C$$

for some $C > 0$ *depending on* Ω, n, p, $\|\mathbf{b}\|_{L^{\infty}(\Omega)}$, $\|c\|_{L^p(\Omega)}$, φ *and* ψ.

We will prove Theorem 1.2 in Section 5. Furthermore, in two dimensions, when $\|\mathbf{b}\|_{L^\infty(\Omega)}$ is small, depending on Ω, ψ and ψ, the conclusions of Theorem 1.2 still hold; see Remark 5.5.

The lack of nonpositivity of c in (1.6) can raise more difficulties in the L^∞ estimate and in the use of the Legendre transform in the Hessian determinant estimates. Compared to the weakest assumption $c \in L^p(\Omega)$ with $p > n$ in [18], we need $p > 2n$ in Theorem 1.2 (ii). However, in two dimensions, this assumption can be weakened provided stronger conditions on $\mathbf{b}$ are imposed, but $\|\mathbf{b}\|_{L^\infty(\Omega)}$ can be arbitrarily large. This is the content of our final main result.

Theorem 1.3 (Solvability of the second boundary value problem for singular Abreu equations with lower order terms in two dimensions). *Let $\Omega \subset \mathbb{R}^2$ be an open, smooth, bounded and uniformly convex domain. Assume that $\varphi \in C^5(\overline{\Omega})$ and $\psi \in C^3(\overline{\Omega})$ with $\min_{\partial\Omega} \psi > 0$. Consider the second boundary value problem (1.6). Assume that $\mathbf{b} \in C^1(\overline{\Omega}; \mathbb{R}^n)$ with $\operatorname{div}(\mathbf{b}) \leq 32/\operatorname{diam}(\Omega)^2$, and $c(x,z) \equiv c(x) \in L^p(\Omega)$ with $p > 2$. Then there exists a uniformly convex solution $u \in W^{4,p}(\Omega)$ to (1.6) with*

$$\|u\|_{W^{4,p}(\Omega)} \leq C$$

for some $C > 0$ depending on Ω, p, $\mathbf{b}$, $\|c\|_{L^p(\Omega)}$, φ and ψ.

The proof of Theorem 1.3 will be given in Section 6.

Remark 1.4. Some remarks are in order.

(1) Theorem 1.1 applies to all convex functions $F(x) = |x|^q/q$ $(q > 1)$ on $\mathbb{R}^n$ for which (1.5) becomes (1.1) when $\mathbf{b} = 0$. Note that if $1 < q < 2$, then $|x|^q \in W^{2,r}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $n < r < n/(2-q)$, while if $q \geq 2$, we have $|x|^q \in W^{2,r}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $r > n$.

(2) By the Sobolev embedding theorem, the solutions u obtained in our main results belong at least to $C^{3,\beta}(\overline{\Omega})$ for some $\beta > 0$.

(3) The condition $\operatorname{div}(\mathbf{b}) \leq 32/\operatorname{diam}(\Omega)^2$ in Theorem 1.3 is due to the method of its proof in obtaining a priori L^∞ estimates that uses a Poincaré type inequality on planar convex domains in Lemma 6.2.

Remark 1.5. We briefly relate the hypotheses in our existence results to concrete examples in applications.

(1) Theorem 1.1 applies to the approximation problem of the variational problem

$$\inf \int_\Omega F_0(x, u(x), Du(x))\, dx \tag{1.7}$$

among certain classes of convex competitors, say, with the same boundary value φ on $\partial\Omega$, where $F_0(x,z,\mathbf{p}) = F(\mathbf{p}) + \hat{F}(x,z)$ with F being convex and $c(x,z) \equiv \frac{\partial \hat{F}}{\partial z}(x,z) \leq 0$. The case $F \equiv 0$ is applicable. One particular example is $F_0(x,z,\mathbf{p}) = \hat{F}(x,z) = (|x|^2/2 - z)\det D^2 v(x)$ where v is a given function, which arises in wrinkling patterns in floating elastic shells in elasticity [36].

(2) Consider now $F_0(x,z,\mathbf{p}) = \hat{F}(x,z)$. Denote $c(x,z) \equiv \frac{\partial \hat{F}}{\partial z}(x,z)$. We note that without the condition $c(x,z) \le 0$, (1.7) might not have a minimizer. (For example, if $\hat{F}(x,z) = z^3$ so $c(x,z) = 3z^2 \ge 0$, then the infimum value of (1.7) is $-\infty$ if $\varphi \not\equiv 0$.) On the other hand, when the assumption $c(x,z) \le 0$ holds, a solution to (1.7) always exists: One solution is the maximal convex extension of φ from $\partial\Omega$ to Ω. The existence results in Theorem 1.1 imply that when $\hat{F}(x,z)$ is perturbed by convex functions of Du (such as $F(Du)$ where F is convex) and $\det D^2 u$ (such as $-\log\det D^2 u$), critical points of the resulting functionals, under appropriate boundary conditions, always exist, and this heuristically means that the resulting functionals continue to have minimizers.

(3) Theorem 1.2 applies to (1.6) with right-hand side $-\Delta u + n + 1$. This expression arises from the Rochet–Choné model with quadratic cost $F_0(x,z,\mathbf{p}) = |\mathbf{p}|^2/2 - x\cdot\mathbf{p} + z$, due to

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial F_0}{\partial p_i}(x,u,Du)\right) + \frac{\partial F_0}{\partial z}(x,u,Du) = -\Delta u + n + 1.$$

Remark 1.6. Given our existence results concerning (1.1), one might wonder if the solutions found are unique. In general, for fourth-order equations, we cannot obtain the uniqueness of solutions by using the comparison principle. However, for equations of the type (1.1), we can obtain uniqueness in some special cases by exploring their particular structure, using integral methods, and taking into account the concavity of the operator $\log\det D^2 u$ and the convexity of $|x|^q/q$ ($q > 1$) or F in general. For example, we can infer from the arguments in [22, Lemma 4.5] that uniqueness holds for (1.1) when $\mathbf{b} \equiv 0$ and $c(x,z)$ satisfies the following monotonicity condition:

$$(c(x,z) - c(x,\tilde{z}))(z-\tilde{z}) \ge 0 \quad \text{for all } x \in \overline{\Omega} \text{ and } z,\tilde{z} \in \mathbb{R}.$$

In particular, this implies that the solutions in Theorem 1.1 (ii) are unique, and the solutions in Theorems 1.2 and 1.3 are unique provided that $\mathbf{b} \equiv 0$. To the best of our knowledge, the uniqueness for (1.1) when $\mathbf{b} \ne 0$ is an interesting open issue.

1.3. On the proofs of the main results

Let us now say a few words about the proofs of our main results using a priori estimates and degree theory. We focus on the most crucial point that overcomes the obstacles encountered in previous works: obtaining an a priori Hölder estimate for $w = (\det D^2 u)^{-1}$ in higher dimensions, once Hessian determinant bounds on u have been obtained. In this case, global Hölder estimates for Du follow. Here, we use a new equivalent form (see Lemma 2.1) for the singular Abreu equation to deal with the difficulties mentioned in Section 1.1. In particular, in Theorem 1.1, instead of establishing a Hölder estimate for w, we establish a Hölder estimate for $\eta = we^{F(Du)}$. The key observation is that η solves a linearized Monge–Ampère equation with a drift term in which the very singular term

$$\operatorname{div}(DF(Du)) = \operatorname{trace}(D^2 F(Du) D^2 u)$$

no longer appears. Thus, the proof of Theorem 1.1 reduces global higher order derivative estimates for (1.5) to global Hölder estimates for linearized Monge–Ampère equations with drift terms. To the best of the authors' knowledge, such global Hölder estimates in full generality are not available in the literature. In the case of Theorems 1.2 and 1.3, the drift terms are also Hölder continuous. However, they do not vanish on the boundary and it seems to be difficult to prove Hölder estimates for η at the boundary, not to mention global Hölder estimates. We overcome this difficulty by observing that *each of our singular Abreu equations is in fact equivalent to a family of linearized Monge–Ampère equations with drifts*. In particular, at each boundary point x_0,

$$\eta^{x_0}(x) = w(x)e^{F(Du(x))-DF(Du(x_0))\cdot(Du(x)-Du(x_0))-F(Du(x_0))}$$

solves a linearized Monge–Ampère equation with a Hölder continuous drift that vanishes at x_0. This gives pointwise Hölder estimates for η^{x_0} (and hence for η) at x_0. Combining this with interior Hölder estimates for linearized Monge–Ampère equations with bounded drifts, we obtain global Hölder estimates for η and hence for w. Section 3 will discuss all these in detail.

For the reader's convenience, we recall the following notion of pointwise Hölder continuity.

Definition 1.7 (Pointwise Hölder continuity). A continuous function $v \in C(\overline{\Omega})$ is said to be *pointwise* C^α $(0 < \alpha < 1)$ at a boundary point $x_0 \in \partial\Omega$, if there exist constants $\delta, M > 0$ such that

$$|v(x) - v(x_0)| \leq M|x - x_0|^\alpha \quad \text{for all } x \in \Omega \cap B_\delta(x_0).$$

Throughout, we use the convention that repeated indices are summed.

The paper is organized as follows. In Section 2, we establish a new equivalent form for singular Abreu equations by transform them into linearized Monge–Ampère equations with drift terms, and the dual equations under the Legendre transform. Global Hölder estimates for linearized Monge–Ampère equations with drift terms, under suitable hypotheses, will be addressed in Section 3. With these estimates, we can prove Theorem 1.1 in Section 4. The proofs of Theorems 1.2 and 1.3 will be given in Sections 5 and 6, respectively. In the final Section 7, we discuss (1.1) with more general lower order terms, and present a proof of Theorem 3.2 on global Hölder estimates for solutions to linearized Monge–Ampère equations with a drift term that are pointwise Hölder continuous at the boundary.

2. Equivalent forms of singular Abreu equations

In this section, we derive some equivalent forms for the following general singular Abreu equations:

$$\begin{cases} U^{ij}D_{ij}w = -\operatorname{div}(DF(Du)) + Q(x,u,Du) & \text{in } \Omega, \\ w = (\det D^2u)^{-1} & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $U = (U^{ij}) = (\det D^2u)(D^2u)^{-1}$, $F \in W^{2,n}_{\mathrm{loc}}(\mathbb{R}^n)$, and Q is a function on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$.

2.1. Singular Abreu equations and linearized Monge–Ampère equations with drifts

Our key observation is the following lemma.

Lemma 2.1 (Equivalence of singular Abreu equations and linearized Monge–Ampère equations with drifts). *Assume that a locally uniformly convex function $u \in W^{4,s}_{\mathrm{loc}}(\Omega)$ ($s > n$) solves (2.1). Then*

$$\eta = we^{F(Du)}$$

satisfies

$$U^{ij}D_{ij}\eta - (\det D^2u)DF(Du)\cdot D\eta = e^{F(Du)}Q(x,u,Du). \tag{2.2}$$

Proof. Let $(u^{ij}) = (D^2u)^{-1} = wU$. By computations using $D_jU^{ij} = 0$ and $w = (\det D^2u)^{-1}$, we have

$$U^{ij}D_{ij}w = D_j(U^{ij}D_iw) = D_j(u^{ij}D_i(\log w)) = -D_j(u^{ij}D_i(\log\det D^2u))$$

and

$$D_j[u^{ij}D_i(F(Du))] = \operatorname{div}(DF(Du)). \tag{2.3}$$

It follows that equation (2.1) can be written as

$$D_j(u^{ij}D_i\zeta) = -Q(x,u,Du) \tag{2.4}$$

where

$$\zeta = \log\det D^2u - F(Du).$$

In other words, in (2.1), the singular term

$$\operatorname{div}(DF(Du)) = \operatorname{trace}(D^2F(Du)D^2u)$$

can be absorbed into the left-hand side to turn it into a divergence form equation.

Next, observe that $\zeta = -\log\eta$, and

$$D_i\zeta = -D_i\eta/\eta = -D_i\eta(\det D^2u)e^{-F(Du)}.$$

Thus (2.4) becomes

$$\begin{aligned}
Q(x,u,Du) &= -D_j(u^{ij}D_i\zeta)\\
&= D_j(u^{ij}(\det D^2u)e^{-F(Du)}D_i\eta)\\
&= D_j(U^{ij}e^{-F(Du)}D_i\eta)\\
&= U^{ij}D_j(e^{-F(Du)}D_i\eta) \quad \text{(using the divergence-free property of } (U^{ij}))\\
&= U^{ij}D_{ij}\eta e^{-F(Du)} - U^{ij}D_i\eta e^{-F(Du)}D_kF(Du)D_{kj}u\\
&= U^{ij}D_{ij}\eta e^{-F(Du)} - (\det D^2u)e^{-F(Du)}DF(Du)\cdot D\eta.
\end{aligned}$$

Therefore, (2.2) holds, and the lemma is proved. ∎

Remark 2.2. In general, (2.1) is not the Euler–Lagrange equation of any functional. However, the introduction of

$$\eta = we^{F(Du)} = (\det D^2 u)^{-1} e^{F(Du)}$$

in Lemma 2.1 has its root in an energy functional. Indeed, when $Q \equiv 0$, (2.1) becomes

$$D_{ij}(U^{ij}(\det D^2 u)^{-1}) + \operatorname{div}(DF(Du)) = 0,$$

and this is the Euler–Lagrange equation of the Monge–Ampère type functional

$$\int_\Omega \big(F(Du) - \log\det D^2 u\big)\, dx = \int_\Omega \log\big((\det D^2 u)^{-1} e^{F(Du)}\big)\, dx.$$

Remark 2.3. Taking $F(x) = |x|^q/q$ with $q > 1$ in Lemma 2.1 where $x \in \mathbb{R}^n$, we find that an equivalent form of

$$U^{ij} D_{ij} w = -\operatorname{div}(|Du|^{q-2} Du) + Q(x, u, Du), \quad w = (\det D^2 u)^{-1},$$

is

$$U^{ij} D_{ij}\eta - (\det D^2 u)|Du|^{q-2} Du \cdot D\eta = Q(x, u, Du) e^{|Du|^q/q}, \tag{2.5}$$

where

$$\eta = we^{|Du|^q/q}.$$

Lemma 2.1 shows that $\eta = we^{F(Du)}$, where u is a solution of (2.1), satisfies a linearized Monge–Ampère equation with a drift term. This fact plays a crucial role in the study of singular Abreu equations in higher dimensions. Once we have estimates for $\det D^2 u$ for the second boundary value problem of (2.1), we can estimate u in $C^{1,\alpha}(\overline{\Omega})$ provided the boundary data is smooth. This gives nice regularity properties for the right-hand side of (2.2) (and in particular (2.5)) and the drift on the left-hand side. Then higher regularity estimates for (2.1) can be reduced to global Hölder estimates for the following linearized Monge–Ampère equation with a drift term:

$$U^{ij} D_{ij}\eta + \mathbf{b} \cdot D\eta + f(x) = 0. \tag{2.6}$$

This is the content of Section 3.

2.2. *Singular Abreu equations under the Legendre transform*

In this section, we derive the dual equation of (2.1) under the Legendre transform in any dimension. After performing the Legendre transform, the dual equation is still a linearized Monge–Ampère equation.

Define the Legendre transform u^* of u by

$$u^*(y) = x \cdot Du - u, \quad \text{where} \quad y = Du(x) \in \Omega^* = Du(\Omega).$$

Then

$$x = Du^*(y) \quad \text{and} \quad u(x) = y \cdot Du^*(y) - u^*(y).$$

Proposition 2.4 (Dual equations for singular Abreu equations). *Let $u \in W^{4,s}_{\text{loc}}(\Omega)$ $(s > n)$ be a uniformly convex solution to (2.1) in Ω. Then in $\Omega^* = Du(\Omega)$, its Legendre transform u^* satisfies the following dual equation:*

$$u^{*ij} D_{ij}(w^* + F(y)) = Q(Du^*, y \cdot Du^* - u^*, y). \tag{2.7}$$

Here (u^{*ij}) is the inverse matrix of $D^2 u^*$, and $w^* = \log\det D^2 u^*$.

Proof. When (2.1) is the Euler–Lagrange equation of a Monge–Ampère type functional, we can derive its dual equation from the dual functional as in [28, Proposition 2.1]. Here for the general case, we prove it by direct calculations. Note that when the right-hand side has no singular term, the dual equation has been obtained in [18, Lemma 2.7]. We include a complete proof here for the reader's convenience.

For simplicity, let $d = \det D^2 u$ and $d^* = \det D^2 u^*$. Then $d(x) = d^{*-1}(y)$ where $y = Du(x)$. We will simply write $d = d^{*-1}$ with this understanding.

We denote by (u^{ij}) and (u^{*ij}) the inverses of the Hessian matrices $D^2 u = (D_{ij} u) = (\frac{\partial^2 u}{\partial x_i \partial x_j})$ and $D^2 u^* = (D_{ij} u^*) = (\frac{\partial^2 u^*}{\partial y_i \partial y_j})$, respectively. Let $(U^{*ij}) = (\det D^2 u^*)(u^{*ij})$ be the cofactor matrix of $D^2 u^*$.

Note that $w = d^{-1} = d^*$. Thus

$$\begin{aligned} D_j w = \frac{\partial w}{\partial x_j} &= \frac{\partial d^*}{\partial y_k}\frac{\partial y_k}{\partial x_j} = \frac{\partial d^*}{\partial y_k} D_{kj} u \\ &= \frac{\partial d^*}{\partial y_k} u^{*kj}. \end{aligned}$$

Clearly,

$$d^{*-1}\frac{\partial d^*}{\partial y_k} = \frac{\partial}{\partial y_k}(\log d^*) = D_{y_k} w^*,$$

from which it follows that

$$D_{x_j} w = D_{y_k} w^* (U^*)^{kj}.$$

Similarly,

$$D_{ij} w = \left(\frac{\partial}{\partial y_l} D_j w\right) u^{*li}.$$

Hence, using

$$U^{ij} = \det D^2 u \cdot u^{ij} = (d^*)^{-1} D_{y_i y_j} u^*,$$

and the fact that $U^* = (U^{*ij})$ is divergence-free, we obtain

$$\begin{aligned} U^{ij} D_{ij} w = (d^*)^{-1} D_{y_i y_j} u^* u^{*li} \frac{\partial}{\partial y_l} D_j w &= (d^*)^{-1}\left(\frac{\partial}{\partial y_j} D_j w\right) \\ &= (d^*)^{-1}\frac{\partial}{\partial y_j}(D_{y_k} w^* U^{*kj}) = (d^*)^{-1} U^{*kj} D_{y_k y_j} w^* \\ &= u^{*ij} D_{ij} w^*. \end{aligned} \tag{2.8}$$

On the other hand, by (2.3), we have

$$\begin{aligned}\operatorname{div}(DF(Du)) = D_{x_j}[u^{ij}D_{x_i}(F(Du))] &= u^{*lj}\frac{\partial}{\partial y_l}\left[u^*_{ij}u^{*ki}\frac{\partial}{\partial y_k}(F(y))\right]\\ &= u^{*ij}D_{y_iy_j}(F(y)).\end{aligned} \tag{2.9}$$

Combining (2.8) with (2.9) and recalling (2.1), we obtain

$$u^{*ij}D_{ij}(w^* + F(y)) = Q(x, u(x), Du(x)) = Q(Du^*, y\cdot Du^* - u^*, y),$$

which is (2.7). The lemma is proved. ■

3. Hölder estimates for linearized Monge–Ampère equation with drifts

In this section, we study global Hölder estimates for the linearized Monge–Ampère equation with drift

$$\begin{cases} U^{ij}D_{ij}v + \mathbf{b}\cdot Dv = f & \text{in } \Omega,\\ v = \varphi & \text{on } \partial\Omega,\end{cases} \tag{3.1}$$

where $U = (U^{ij}) = (\det D^2u)(D^2u)^{-1}$ and $\mathbf{b} : \Omega \to \mathbb{R}^n$ is a vector field.

When there is no drift term, that is, $\mathbf{b} \equiv 0$, global Hölder estimates for (3.1) were established under suitable assumptions on the bounds $0 < \lambda \le \det D^2u \le \Lambda$ on the Hessian determinant of u, and the data. In particular, the case $f \in L^n(\Omega)$ was treated in [17, Theorem 1.4] (see also [25, Theorem 4.1] for a more localized version) and the case $f \in L^{n/2+\varepsilon}(\Omega)$ was dealt with in [26, Theorem 1.7].

We wish to extend the above global Hölder estimates to the case with bounded drift. In this case, the interior Hölder estimates for (3.1) were obtained as a consequence of the interior Harnack inequality proved in [20, Theorem 1.1]. Note that Maldonado [29] also proved a Harnack inequality for (3.1) with different and stronger conditions on $\mathbf{b}$.

Therefore, to obtain global Hölder estimates for (3.1) with a bounded drift $\mathbf{b}$, it remains to prove Hölder estimates at the boundary. Without further assumptions on $\mathbf{b}$, this seems to be difficult with current techniques. However, when $\mathbf{b}$ is pointwise Hölder continuous, and vanishes at a boundary point x_0, we can obtain the pointwise Hölder continuity of v at x_0. This can be deduced from the following result, which is a drift version of [17, Proposition 2.1].

Proposition 3.1 (Pointwise Hölder estimate at the boundary for solutions to nonuniformly elliptic, linear equations with pointwise Hölder continuous drift). *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded, uniformly convex domain. Let $\varphi \in C^\alpha(\partial\Omega)$ for some $\alpha \in (0,1)$, and $g \in L^n(\Omega)$. Assume that the matrix (a^{ij}) is measurable, positive definite and satisfies $\det(a^{ij}) \ge \lambda$ in Ω. Let $\mathbf{b} \in L^\infty(\Omega;\mathbb{R}^n)$. Let $v \in C(\overline{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ be a solution to*

$$a^{ij}D_{ij}v + \mathbf{b}\cdot Dv = g \text{ in } \Omega, \quad v = \varphi \text{ on } \partial\Omega.$$

Suppose there are constants $\mu, \tau \in (0,1)$ *and* $M > 0$ *such that at some* $x_0 \in \partial\Omega$, *we have*

$$|\mathbf{b}(x)| \leq M|x - x_0|^{\mu} \quad \textit{for all } x \in \Omega \cap B_\tau(x_0). \tag{3.2}$$

Then there exist δ, C *depending only on* $\lambda, n, \alpha, \mu, \tau, M$, $\|\mathbf{b}\|_{L^\infty(\Omega)}$ *and* Ω *such that*

$$|v(x) - v(x_0)| \leq C|x - x_0|^{\frac{\min\{\alpha,\mu\}}{\min\{\alpha,\mu\}+4}}(\|\varphi\|_{C^\alpha(\partial\Omega)} + \|g\|_{L^n(\Omega)}) \textit{ for all } x \in \Omega \cap B_\delta(x_0).$$

We will prove Proposition 3.1 in Section 3.1.

Once we have the pointwise Hölder estimates at the boundary, global Hölder estimates for (3.1) follow. This is the content of the following theorem.

Theorem 3.2 (Global Hölder estimates for solutions to the linearized Monge–Ampère equation with a drift term that are pointwise Hölder continuous at the boundary). *Assume that* $\Omega \subset \mathbb{R}^n$ *is a uniformly convex domain with* $\partial\Omega \in C^3$. *Let* $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ *be a convex function satisfying*

$$\lambda \leq \det D^2 u \leq \Lambda \quad \textit{in } \Omega$$

for some positive constants λ *and* Λ. *Moreover, assume that* $u|_{\partial\Omega} \in C^3$. *Let* $(U^{ij}) = (\det D^2 u)(D^2 u)^{-1}$. *Let* $\mathbf{b} \in L^\infty(\Omega; \mathbb{R}^n)$ *with* $\|\mathbf{b}\|_{L^\infty(\Omega)} \leq M$, $f \in L^n(\Omega)$ *and* $\varphi \in C^\alpha(\partial\Omega)$ *for some* $\alpha \in (0,1)$. *Assume that* $v \in C(\overline{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ *is a solution to the following linearized Monge–Ampère equation with a drift term:*

$$\begin{cases} U^{ij} D_{ij} v + \mathbf{b} \cdot Dv = f & \textit{in } \Omega, \\ v = \varphi & \textit{on } \partial\Omega. \end{cases}$$

Suppose that there exist $\gamma \in (0, \alpha]$, $\delta > 0$ *and* $K > 0$ *such that*

$$|v(x) - v(x_0)| \leq K|x - x_0|^{\gamma} \quad \textit{for all } x_0 \in \partial\Omega \textit{ and } x \in \Omega \cap B_\delta(x_0). \tag{3.3}$$

Then there exist a constant $\beta \in (0,1)$ *depending on* n, λ, Λ, γ *and* M, *and a constant* $C > 0$ *depending only on* Ω, $u|_{\partial\Omega}$, λ, Λ, n, α, γ, δ, K *and* M, *such that*

$$|v(x) - v(y)| \leq C|x - y|^{\beta}(\|\varphi\|_{C^\alpha(\partial\Omega)} + \|f\|_{L^n(\Omega)}) \quad \textit{for all } x, y \in \Omega.$$

The proof of Theorem 3.2 is similar to that of [17, Theorem 1.4] for the case without drift. For completeness and for the reader's covenience, we present the proof in Section 7.

Remark 3.3. It would be interesting to prove the global Hölder estimates in Theorem 3.2 without the assumption (3.3).

In Section 3.2, we will apply Theorem 3.2 to establish global Hölder estimates for Hessian determinants of singular Abreu equations provided that the Hessian determinants are bounded between two positive constants; see Theorem 3.4.

3.1. Pointwise Hölder estimates at the boundary

In this section, we prove Proposition 3.1.

Proof of Proposition 3.1. The proof is similar to that of [17, Proposition 2.1]. Due to the appearance of the drift $\mathbf{b}$ and the pointwise Hölder continuity condition (3.2), we include the proof for the reader's convenience.

Let

$$K = \|\mathbf{b}\|_{L^\infty(\Omega)}, \quad L = \operatorname{diam}(\Omega).$$

In this proof, we fix the exponent

$$\gamma = \min\{\alpha, \mu\}/2.$$

However, the proof works for any exponent $\gamma \in (0,1)$ such that $\gamma < \min\{\alpha, \mu\}$, and in this case, we replace the exponent $\frac{\min\{\alpha,\mu\}}{\min\{\alpha,\mu\}+4}$ in the proposition by $\frac{\gamma}{\gamma+2}$.

Clearly $\varphi \in C^\gamma(\partial\Omega)$ with $\|\varphi\|_{C^\gamma(\partial\Omega)} \leq C(\alpha, \mu, L)\|\varphi\|_{C^\alpha(\partial\Omega)}$. By considering the equation satisfied by $(\|\varphi\|_{C^\gamma(\partial\Omega)} + \|g\|_{L^n(\Omega)})^{-1} v$, we can assume that

$$\|\varphi\|_{C^\gamma(\partial\Omega)} + \|g\|_{L^n(\Omega)} = 1,$$

and it suffices to prove that, for some $\delta = \delta(n, \lambda, \alpha, \tau, K, M, \mu, \Omega) > 0$, we have

$$|v(x) - v(x_0)| \leq C(n, \lambda, \alpha, \tau, K, M, \mu, \Omega)|x - x_0|^{\frac{\gamma}{\gamma+2}} \quad \text{for all } x \in \Omega \cap B_\delta(x_0).$$

Moreover, without loss of generality, we assume that

$$\Omega \subset \mathbb{R}^n \cap \{x_n > 0\}, \quad x_0 = 0 \in \partial\Omega.$$

Since $\det(a^{ij}) \geq \lambda$, by the Aleksandrov–Bakelman–Pucci (ABP) estimate for elliptic, linear equations with drift (see [14, inequality (9.14)]), we have

$$\begin{aligned}\|v\|_{L^\infty(\Omega)} &\leq \|\varphi\|_{L^\infty(\partial\Omega)} + \operatorname{diam}(\Omega)\left\{\exp\left[\frac{2^{n-2}}{n^n\omega_n}\int_\Omega\left(1 + \frac{|\mathbf{b}|^n}{\det(a^{ij})}\right)dx\right] - 1\right\}^{1/n} \\ &\qquad \times \left\|\frac{g}{(\det(a^{ij}))^{1/n}}\right\|_{L^n(\Omega)} \\ &\leq C_0 \end{aligned} \tag{3.4}$$

for a constant $C_0(n, \lambda, K, L) > 1$. Here we have used $\omega_n = |B_1(0)|$ and $\|\varphi\|_{C^\gamma(\partial\Omega)} + \|g\|_{L^n(\Omega)} = 1$. Hence, for any $\varepsilon \in (0, \tau^\gamma)$,

$$|v(x) - v(0) \pm \varepsilon| \leq 3C_0 =: C_1. \tag{3.5}$$

Consider now the functions

$$\psi_\pm(x) := v(x) - v(0) \pm \varepsilon \pm C_1\kappa(\delta_2)x_n,$$

where

$$\kappa(\delta_2) := (\inf\{y_n : y \in \overline{\Omega} \cap \partial B_{\delta_2}(0)\})^{-1}$$

in the region

$$A := \Omega \cap B_{\delta_2}(0),$$

where $\delta_2 < 1$ is small to be chosen later.

The uniform convexity of Ω gives

$$\inf\{y_n : y \in \overline{\Omega} \cap \partial B_{\delta_2}(0)\} \geq C_2^{-1}\delta_2^2, \tag{3.6}$$

where C_2 depends on the uniform convexity of Ω. Thus,

$$\kappa(\delta_2) \leq C_2\delta_2^{-2}.$$

Note that if $x \in \partial\Omega$ with $|x| \leq \delta_1(\varepsilon) := \varepsilon^{1/\gamma}$ $(\leq \tau)$, then from $\|\varphi\|_{C^\gamma(\partial\Omega)} \leq 1$ we have

$$|v(x) - v(0)| = |\varphi(x) - \varphi(0)| \leq |x|^\gamma \leq \varepsilon. \tag{3.7}$$

It follows that if we choose $\delta_2 \leq \delta_1$, then from (3.5) and (3.7) we have

$$\psi_- \leq 0, \quad \psi_+ \geq 0 \quad \text{on } \partial A.$$

From (3.2) we have

$$|\mathbf{b}| \leq M\delta_2^\mu \quad \text{in } A,$$

and therefore

$$a^{ij}D_{ij}\psi_- + \mathbf{b}\cdot D\psi_- = g - C_1\kappa(\delta_2)\mathbf{b}\cdot e_n \geq -|g| - C_1C_2M\delta_2^{\mu-2} \quad \text{in } A,$$

where $e_n = (0, \ldots, 0, 1) \in \mathbb{R}^n$.

Similarly,

$$a^{ij}D_{ij}\psi_+ + \mathbf{b}\cdot D\psi_+ = g + C_1\kappa(\delta_2)\mathbf{b}\cdot e_n \leq |g| + C_1C_2M\delta_2^{\mu-2} \quad \text{in } A.$$

Again, applying the ABP estimate for elliptic, linear equations with drift, we obtain

$$\begin{aligned}\psi_- &\leq C(n, \lambda, K, L)\,\mathrm{diam}(A)\|g + C_1C_2M\delta_2^{\mu-2}\|_{L^n(A)}\\ &\leq C_3(n, \lambda, K, M, \Omega, \tau, \mu)\delta_2^\mu \quad \text{in } A.\end{aligned}$$

In the above inequality, we have used $\|g\|_{L^n(A)} \leq 1$ and

$$\begin{aligned}\|g + C_1C_2M\delta^{\mu-2}\|_{L^n(A)} &\leq \|g\|_{L^n(A)} + C_1C_2M\delta_2^{\mu-2}|A|^{1/n}\\ &\leq C(n, \lambda, K, M, \Omega, \tau, \mu)\delta_2^{\mu-1}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\psi_+ &\geq -C(n, \lambda, K, L)\,\mathrm{diam}(A)\|g + C_1C_2M\delta_2^{\mu-2}\|_{L^n(A)}\\ &\geq -C_3(n, \lambda, K, M, \Omega, \tau, \mu)\delta_2^\mu \quad \text{in } A.\end{aligned}$$

We now restrict $\varepsilon \leq C_3^{-\gamma/(\mu-\gamma)}$ so that

$$\delta_1 = \varepsilon^{1/\gamma} \leq [\varepsilon/C_3]^{1/\mu}.$$

Then, for $\delta_2 \leq \delta_1$, we have $C_3\delta_2^\mu \leq \varepsilon$, and thus

$$|v(x) - v(0)| \leq 2\varepsilon + C_1\kappa(\delta_2)x_n \quad \text{in } A.$$

Therefore, choosing $\delta_2 = \delta_1$, we find

$$|v(x) - v(0)| \le 2\varepsilon + C_1\kappa(\delta_2)x_n \le 2\varepsilon + \frac{2C_1C_2}{\delta_2^2}x_n \quad \text{in } A.$$

Summarizing, we obtain the inequality

$$|v(x) - v(0)| \le 2\varepsilon + \frac{2C_1C_2}{\delta_2^2}|x| \le 2\varepsilon + 2C_1C_2\varepsilon^{-2/\gamma}|x| \tag{3.8}$$

for all x, ε satisfying

$$|x| \le \delta_1(\varepsilon) := \varepsilon^{1/\gamma}, \quad \varepsilon \le C_3^{-\gamma/(\mu-\gamma)} =: c_1. \tag{3.9}$$

Let us now choose $\varepsilon = |x|^{\gamma/(\gamma+2)}$. Then the conditions in (3.9) are satisfied as long as

$$|x| \le \min\{c_1^{(\gamma+2)/\gamma}, 1\} =: \delta.$$

With this choice of δ, and recalling (3.8), we have

$$|v(x) - v(0)| \le (2 + 2C_1C_2)|x|^{\gamma/(\gamma+2)} \quad \text{for all } x \in \Omega \cap B_\delta(0).$$

The proposition is proved. ∎

3.2. Singular Abreu equations with Hessian determinant bounds

In this section, we apply Theorem 3.2 to establish global Hölder estimates for Hessian determinants of singular Abreu equations provided that the Hessian determinants are bounded between two positive constants. This is the content of the following theorem.

Theorem 3.4 (Hölder continuity of Hessian determinant of singular Abreu equations under Hessian determinant bounds). *Assume that $\Omega \subset \mathbb{R}^n$ is a uniformly convex domain with $\partial\Omega \in C^3$. Let $F \in W^{2,r}_{\text{loc}}(\mathbb{R}^n)$ for some $r > n$, and let $g \in L^s(\Omega)$ with $s > n$. Let $\varphi \in C^4(\overline{\Omega})$ and $\psi \in C^2(\overline{\Omega})$ with $\min_{\partial\Omega}\psi > 0$. Assume that $u \in W^{4,s}(\Omega)$ is a uniformly convex solution to the singular Abreu equation*

$$\begin{cases} U^{ij}D_{ij}w = -\operatorname{div}(DF(Du)) + g(x) & \textit{in } \Omega, \\ w = (\det D^2u)^{-1} & \textit{in } \Omega, \\ u = \varphi & \textit{on } \partial\Omega, \\ w = \psi & \textit{on } \partial\Omega, \end{cases}$$

where $U = (U^{ij}) = (\det D^2u)(D^2u)^{-1}$. Suppose that, for some positive constants λ and Λ,

$$\lambda \le \det D^2u \le \Lambda \quad \textit{in } \Omega.$$

Then there exist constants $\beta, C > 0$ depending only on Ω, φ, ψ, λ, Λ, n, r, F and $\|g\|_{L^n(\Omega)}$ such that

$$\|w\|_{C^\beta(\overline{\Omega})} \le C.$$

Proof. Since $F \in W^{2,r}_{\mathrm{loc}}(\mathbb{R}^n)$, by the Sobolev embedding theorem we have $F \in C^{1,\alpha}(\mathbb{R}^n)$ where $\alpha = 1 - n/r \in (0,1)$. From the Hessian determinant bounds on u, and $u = \varphi$ on $\partial\Omega$ where $\varphi \in C^4(\bar{\Omega})$, by [27, Proposition 2.6] we have

$$\|u\|_{C^{1,\alpha_0}(\bar{\Omega})} \leq C_1, \tag{3.10}$$

where $\alpha_0 \in (0,1)$ depends on λ, Λ and n. The constant C_1 depends on $\Omega, n, \lambda, \Lambda$ and φ.

By Lemma 2.1, the function

$$\eta(x) = w(x)e^{F(Du(x))}$$

satisfies

$$U^{ij}D_{ij}\eta - (\det D^2u)DF(Du(x)) \cdot D\eta = g(x)e^{F(Du(x))} =: f(x). \tag{3.11}$$

From (3.10), we deduce that $\eta|_{\partial\Omega} \in C^{\alpha_0}$ with estimate

$$\|\eta\|_{C^{\alpha_0}(\partial\Omega)} \leq C_*(\psi, C_1, F). \tag{3.12}$$

Step 1: Pointwise Hölder continuity of η at the boundary. Fix $x_0 \in \partial\Omega$ and denote

$$\tilde{F}(y) := F(y) - F(Du(x_0)) - DF(Du(x_0)) \cdot (y - Du(x_0)) \quad \text{for } y \in \mathbb{R}^n.$$

Then

$$U^{ij}D_{ij}w(x) = -\operatorname{div}(D\tilde{F}(Du(x))) + g(x) \quad \text{in } \Omega.$$

By Lemma 2.1, the function

$$\eta^{x_0}(x) = w(x)e^{\tilde{F}(Du(x))}$$

satisfies

$$\begin{aligned} U^{ij}D_{ij}\eta^{x_0} - (\det D^2u)(DF(Du(x)) - DF(Du(x_0))) \cdot D\eta^{x_0} \\ = g(x)e^{\tilde{F}(Du(x))} =: f^{x_0}(x). \end{aligned} \tag{3.13}$$

Clearly,

$$\|f^{x_0}\|_{L^n(\Omega)} \leq C_2, \tag{3.14}$$

where C_2 depends on $\|F\|_{C^1(B_{C_1}(0))}$ and $\|g\|_{L^n(\Omega)}$.

The vector field

$$\mathbf{b}(x) := (\det D^2u) \cdot \big(DF(Du(x)) - DF(Du(x_0))\big)$$

satisfies in Ω the estimate

$$\begin{aligned} |\mathbf{b}(x)| &\leq \Lambda \|DF\|_{C^\alpha(B_{C_1}(0))} |Du(x) - Du(x_0)|^\alpha \\ &\leq \Lambda C_1 \|DF\|_{C^\alpha(B_{C_1}(0))} |x - x_0|^{\alpha_1}, \end{aligned} \tag{3.15}$$

where

$$\alpha_1 := \alpha\alpha_0.$$

We also have $\eta^{x_0}|_{\partial\Omega} \in C^{\alpha_1}(\partial\Omega)$ with

$$\|\eta^{x_0}\|_{C^{\alpha_1}(\partial\Omega)} \le C_3(\alpha, \alpha_0, C_1, \psi, \|DF\|_{C^\alpha(B_{C_1}(0))}). \tag{3.16}$$

Note that

$$\det(U^{ij}) = (\det D^2 u)^{n-1} \ge \lambda^{n-1}.$$

Hence, from (3.13), (3.15) and (3.16), we can apply Proposition 3.1 and find constants

$$\gamma = \alpha_1/(\alpha_1 + 4) \in (0, 1),$$

and $\delta, C_4 > 0$ depending only on $n, \lambda, \Lambda, \alpha, F, \varphi, \psi$ and Ω such that, for all $x \in \Omega \cap B_\delta(x_0)$,

$$\begin{aligned} |\eta^{x_0}(x) - \eta^{x_0}(x_0)| &\le C_4|x - x_0|^\gamma (\|\eta^{x_0}\|_{C^{\alpha_1}(\partial\Omega)} + \|f^{x_0}\|_{L^n(\Omega)}) \\ &\le C_5|x - x_0|^\gamma, \end{aligned} \tag{3.17}$$

where $C_5 := C_4(C_2 + C_3)$.

Due to

$$\eta(x) = \eta^{x_0}(x) e^{F(Du(x_0)) + DF(Du(x_0))\cdot(Du(x) - Du(x_0))},$$

and (3.10), inequality (3.17) implies the pointwise C^γ continuity of η at x_0 with estimate

$$|\eta(x) - \eta(x_0)| \le C_6|x - x_0|^\gamma \quad \text{for all } x \in \Omega \cap B_\delta(x_0), \tag{3.18}$$

where C_6 depends on Ω, φ, ψ, λ, Λ, n, α, F and $\|g\|_{L^n(\Omega)}$.

Step 2: Global Hölder continuity of η and w. From (3.18), we can apply Theorem 3.2 to (3.11) to conclude the global Hölder continuity of η. Since $w = \eta e^{-F(Du)}$, w is also globally Hölder continuous. In other words, there exist a constant $\beta \in (0, 1)$ depending on $n, \lambda, \Lambda, \alpha$ and F, and a constant $C > 0$ depending only on Ω, φ, ψ, λ, Λ, n, r, F and $\|g\|_{L^n(\Omega)}$, such that

$$\|w\|_{C^\beta(\bar{\Omega})} \le C.$$

The theorem is proved. ■

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 using a priori estimates and degree theory. With Theorem 3.4 at hand, a key step is to establish a priori Hessian determinant estimates for uniformly convex solutions $u \in W^{4,s}(\Omega)$ $(s > n)$ of (1.5).

To obtain the latter estimates, we will use the maximum principle and the Legendre transform; see also [28, Theorem 1.2] with a slightly different proof for the case of $F(x) = |x|^q/q$ $(q > 1)$ and $c(x, z)$ being smooth.

Lemma 4.1 (Hessian determinant estimates). *Let $\Omega \subset \mathbb{R}^n$ be an open, smooth, bounded and uniformly convex domain. Assume that $\varphi \in C^5(\overline{\Omega})$ and $\psi \in C^3(\overline{\Omega})$ with $\min_{\partial\Omega} \psi > 0$. Let $r, s > n$. Let $F \in W^{2,r}_{\text{loc}}(\mathbb{R}^n)$ be a convex function, and $c(x,z)$ be a function on $\overline{\Omega} \times \mathbb{R}$. Suppose that $c(x,z) \le 0$ with either $c \in C^\alpha(\overline{\Omega} \times \mathbb{R})$ where $\alpha \in (0,1)$, or $c(x,z) \equiv c(x) \in L^s(\Omega)$. Assume that $u \in W^{4,s}(\Omega)$ is a uniformly convex solution to the second boundary value problem*

$$\begin{cases} U^{ij} D_{ij} w = -\operatorname{div}(DF(Du)) + c(x,u) & \text{in } \Omega, \\ w = (\det D^2 u)^{-1} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ w = \psi & \text{on } \partial\Omega, \end{cases}$$

where $(U^{ij}) = (\det D^2 u)(D^2 u)^{-1}$. Then

$$C^{-1} \le \det D^2 u \le \left(\min_{\partial\Omega} \psi\right)^{-1} \quad \text{in } \Omega,$$

where $C > 0$ is a constant depending on Ω, n, φ, ψ, F and c. In the case of $c(x,z) \equiv c(x) \in L^s(\Omega)$, the dependence of C on c is via $\|c\|_{L^n(\Omega)}$.

Proof. From the convexity of F and u, we have

$$-\operatorname{div}(DF(Du)) = -\operatorname{trace}(D^2 F(Du) D^2 u) \le 0.$$

This combined with $c(x,u) \le 0$ yields

$$U^{ij} D_{ij} w = -\operatorname{div}(DF(Du)) + c(x,u) \le 0 \quad \text{in } \Omega.$$

Hence, by the maximum principle, w attains its minimum value in $\overline{\Omega}$ on the boundary. Thus

$$w \ge \min_{\partial\Omega} w = \min_{\partial\Omega} \psi > 0 \quad \text{in } \Omega.$$

This together with $\det D^2 u = w^{-1}$ gives an upper bound for the Hessian determinant:

$$\det D^2 u \le C_1 := \left(\min_{\partial\Omega} \psi\right)^{-1} \quad \text{in } \Omega.$$

From the above upper bound, by using $u = \varphi$ on $\partial\Omega$ together with Ω being smooth and uniformly convex, we can construct suitable barrier functions to deduce that

$$\sup_{\Omega} |u| + \|Du\|_{L^\infty(\Omega)} \le C_2, \tag{4.1}$$

where C_2 depends on n, φ, ψ and Ω.

We now proceed to establish a positive lower bound for the Hessian determinant. Let

$$u^*(y) = x \cdot Du(x) - u(x)$$

be the Legendre transform of $u(x)$, where

$$y = Du(x) \in \Omega^* := Du(\Omega).$$

Then (4.1) implies

$$\operatorname{diam}(\Omega^*) + \|u^*\|_{L^\infty(\Omega^*)} \le C_3(n, \varphi, \psi, \Omega). \tag{4.2}$$

In view of Proposition 2.4, u^* satisfies

$$u^{*ij} D_{ij}(w^* + F(y)) = c(Du^*, y \cdot Du^* - u^*) \quad \text{in } \Omega^*, \tag{4.3}$$

where

$$(u^{*ij}) = (D^2 u^*)^{-1}, \quad w^* = \log\det D^2 u^*.$$

Note that, for $y = Du(x) \in \partial\Omega^*$ where $x \in \partial\Omega$, we have

$$w^*(y) = \log(\det D^2 u(x))^{-1} = \log\psi(x).$$

By the ABP maximum principle applied to (4.3), and recalling (4.2), we find

$$\begin{aligned}
&\sup_{\Omega^*}(w^* + F(y)) \\
&\quad\le \sup_{\partial\Omega^*}(w^* + F(y)) + C(n, \operatorname{diam}(\Omega^*)) \left\| \frac{c(Du^*, y\cdot Du^* - u^*)}{(\det D^2 u^*)^{-1/n}} \right\|_{L^n(\Omega^*)} \\
&\quad= \sup_{\partial\Omega^*}(w^* + F(y)) + C(n, \operatorname{diam}(\Omega^*)) \left(\int_\Omega |c(x,u)|^n \, dx \right)^{1/n} \le C_4,
\end{aligned}$$

where C_4 depends on Ω, n, φ, ψ, F and c. Clearly, in the case of $c(x,z) \equiv c(x) \in L^s(\Omega)$, the dependence of C_4 on c is via $\|c\|_{L^n(\Omega)}$. In the above estimates, we have used

$$\begin{aligned}
\left\| \frac{c(Du^*, y\cdot Du^* - u^*)}{(\det D^2 u^*)^{-1/n}} \right\|_{L^n(\Omega^*)} &= \left(\int_{\Omega^*} |c(Du^*, y\cdot Du^* - u^*)|^n \det D^2 u^* \, dy \right)^{1/n} \\
&= \left(\int_\Omega |c(x,u)|^n \det D^2 u^* \det D^2 u \, dx \right)^{1/n} \\
&= \left(\int_\Omega |c(x,u)|^n \, dx \right)^{1/n}.
\end{aligned}$$

It follows that

$$\sup_{\Omega^*} w^*(y) = \sup_{\Omega^*} \log\det D^2 u^* \le C_5,$$

which implies

$$\det D^2 u \ge e^{-C_5} > 0 \quad \text{in } \Omega,$$

where C_5 depends on Ω, n, φ, ψ, F and c. This is the desired positive lower bound for the Hessian determinant, and the proof of the lemma is complete. ∎

Proof of Theorem 1.1. We divide the proof, using a priori estimates and degree theory, into three steps. Steps 1 and 2 establish higher order derivative estimates for $W^{4,s}(\Omega)$ $(s > n)$ solutions. Step 3 confirms the existence of $W^{4,s}(\Omega)$ or $C^{4,\beta}(\bar{\Omega})$ solutions via degree theory.

In the following, we fix $s > n$ with the additional requirement that

$$\begin{cases} s = r & \text{(case (i))},\\ s = \min\{r, p\} & \text{(case (ii))}. \end{cases}$$

Step 1: Determinant estimates and second order derivative estimates for uniformly convex $W^{4,s}(\Omega)$ $(s > n)$ solutions u of (1.5). By Lemma 4.1, we have

$$0 < \lambda \leq \det D^2 u \leq \Lambda := \Big(\min_{\partial\Omega} \psi\Big)^{-1} \quad \text{in } \Omega, \tag{4.4}$$

where λ depends on $\Omega, n, F, \varphi, \psi$, and either on c in case (i), or on $\|c\|_{L^n(\Omega)}$ in case (ii).

From (4.4) and $u = \varphi$ on $\partial\Omega$ where $\varphi \in C^5(\bar{\Omega})$, by [27, Proposition 2.6] we have

$$\|u\|_{C^{1,\alpha_0}(\bar{\Omega})} \leq C_1, \tag{4.5}$$

where $\alpha_0 \in (0,1)$ depends on λ, Λ, and n. The constant C_1 depends on $\Omega, n, \lambda, \Lambda$ and φ.

With (4.4) and $F \in W^{2,r}_{\text{loc}}(\mathbb{R}^n)$, we can use Theorem 3.4 to find $\beta_0 \in (0,1)$, and $C_2 > 0$ depending on $\Omega, n, F, r, \varphi, \psi, c$, such that

$$\|w\|_{C^{\beta_0}(\bar{\Omega})} \leq C_2(\Omega, n, F, r, \varphi, \psi, c).$$

Hence $\det D^2 u = w^{-1} \in C^{\beta_0}(\bar{\Omega})$. By the global Schauder estimates for the Monge–Ampère equation [34, 39], we have

$$\|u\|_{C^{2,\beta_0}(\bar{\Omega})} \leq C_3(\Omega, n, F, r, \varphi, \psi, c). \tag{4.6}$$

Combining this with (4.4), we find

$$C_4^{-1} I_n \leq D^2 u \leq C_4 I_n \quad \text{in } \Omega$$

for some $C_4(\Omega, n, F, r, \varphi, \psi, c) > 0$. Here I_n denotes the identity $n \times n$ matrix. In other words, the linear operator $U^{ij} D_{ij}$ is uniformly elliptic with coefficients U^{ij} bounded in $C^{\beta_0}(\bar{\Omega})$.

Step 2: Global higher order derivative estimates for uniformly convex $W^{4,s}(\Omega)$ $(s > n)$ solutions u of (1.5). Denote the right-hand side of (1.5) by

$$f := -\operatorname{div}(DF(Du)) + c(x,u) = -\operatorname{trace}(D^2 F(Du) D^2 u) + c(x,u). \tag{4.7}$$

Observe that

$$\|\operatorname{trace}(D^2 F(Du) D^2 u)\|_{L^r(\Omega)} \leq C(\Omega, n, F, r, \varphi, \psi, c). \tag{4.8}$$

Indeed,

$$
\begin{aligned}
\|\mathrm{trace}(D^2F(Du)D^2u)\|^r_{L^r(\Omega)} &\le n^2\|D^2u\|^r_{L^\infty(\Omega)}\|D^2F(Du)\|^r_{L^r(\Omega)}\\
&\le n^2C_3^r\int_\Omega |D^2F(Du(x))|^r\,dx \quad (\text{using } (4.6))\\
&= n^2C_3^r\int_{Du(\Omega)}|D^2F(y)|^r\frac{1}{\det D^2u((Du)^{-1}(y))}\,dy\\
&\le n^2C_3^r\lambda^{-1}\int_{B_{C_1}(0)}|D^2F(y)|^r\,dy\\
&\qquad\qquad (\text{using } (4.4) \text{ and } (4.5))\\
&\le C_3^r\lambda^{-1}C(n,C_1,F,r).
\end{aligned}
$$

We consider cases (i) and (ii) separately.

(i) *The case of* $c \in C^\alpha(\overline{\Omega}\times\mathbb{R})$. Recall that $s = r$ in this case. We see from (4.8) that $f = -\mathrm{trace}(D^2F(Du)D^2u) + c(x,u) \in L^s(\Omega)$ with estimate

$$\|f\|_{L^s(\Omega)} \le C(\Omega,n,F,r,s,\varphi,\psi,c).$$

By Step 1,

$$U^{ij}D_{ij}w = f \quad \text{in } \Omega, \quad w = \psi \quad \text{on } \partial\Omega,$$

is a uniformly elliptic equation in w with $C^{\beta_0}(\overline{\Omega})$ coefficients. Thus, from the standard $W^{2,p}$ theory for uniformly elliptic linear equations (see [14, Chapter 9]), we obtain the following $W^{2,s}(\Omega)$ estimate:

$$\|w\|_{W^{2,s}(\Omega)} \le C(\Omega,n,q,s,\varphi,\psi,c).$$

Now, recalling $\det D^2u = w^{-1}$ in Ω with $u = \varphi$ on $\partial\Omega$, we can differentiate and apply the standard Schauder and Calderón–Zygmund theories to obtain the following global $W^{4,s}$ estimate of u:

$$\|u\|_{W^{4,s}(\Omega)} \le C(\Omega,n,F,r,s,\varphi,\psi,c).$$

Indeed, for any $k \in \{1,\dots,n\}$ by differentiating $\det D^2u = w^{-1}$ in the x_k direction we see that D_ku solves the equation

$$U^{ij}D_{ij}(D_ku) = D_k(w^{-1}) \in W^{1,s}(\Omega),$$

which is uniformly elliptic with $C^{\beta_0}(\overline{\Omega})$ coefficients U^{ij} due to (4.4) and (4.6). Since $s > n$, we have $W^{1,s}(\Omega) \in C^{0,1-n/s}(\overline{\Omega})$. By the classical Schauder theory (see [14, Chapter 6] for example), we deduce that $D_ku \in C^{2,\beta_1}(\overline{\Omega})$ for all k with appropriate estimates, where $\beta_1 = \min\{\beta_0, 1-n/s\}$. This shows that $u \in C^{3,\beta_1}(\overline{\Omega})$ and the coefficients satisfy $U^{ij} \in C^{1,\beta_1}(\overline{\Omega})$. Next, for any $l \in \{1,\dots,n\}$, we differentiate the preceding equation in the x_l direction to get

$$U^{ij}D_{ij}(D_{kl}u) = D_{kl}(w^{-1}) - D_lU^{ij}D_{ijk}u \in L^s(\Omega) \quad \text{for all } k,l \in \{1,\dots,n\}.$$

Applying the Calderón–Zygmund estimates, we obtain $D_{kl}u \in W^{2,s}(\Omega)$ for all $k, l \in \{1, \ldots, n\}$ with appropriate estimates. Consequently, $u \in W^{4,s}(\Omega)$ with estimate stated above.

Moreover, in the particular case that $F \in C^{2,\alpha_0}(\mathbb{R}^n)$, we find that $f \in C^{\gamma}(\overline{\Omega})$ where $\gamma \in (0, 1)$ depends only on α, F, α_0, and β_0 with estimate

$$\|f\|_{C^{\gamma}(\overline{\Omega})} \le C(\Omega, n, \alpha, q, c, \varphi, \psi). \tag{4.9}$$

Thus, we can apply the classical Schauder theory (see [14, Chapter 6] for example) to (1.5) which, by Step 1, is a uniformly elliptic equation in w with $C^{\beta_0}(\overline{\Omega})$ coefficients. We conclude that $w \in C^{2,\beta}(\overline{\Omega})$, where $\beta \in (0, 1)$ depends only on n, γ and β_0, with estimate

$$\|w\|_{C^{2,\beta}(\overline{\Omega})} \le C(\Omega, n, \alpha, \alpha_0, F, c, \varphi, \psi).$$

Due to

$$\det D^2 u = w^{-1} \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega,$$

this implies that $u \in C^{4,\beta}(\overline{\Omega})$ with estimate

$$\|u\|_{C^{4,\beta}(\overline{\Omega})} \le C(\Omega, n, \alpha, \alpha_0, F, c, \varphi, \psi). \tag{4.10}$$

With this estimate, we go back to $f = -\text{trace}(D^2F(Du)D^2u) + c(x, u)$ and find that we can actually take $\gamma = \min\{\alpha, \alpha_0\}$ in (4.9). Repeating the above process, we find that (4.10) holds for $\beta = \min\{\alpha, \alpha_0\}$.

(ii) *The case of $c(x, z) \equiv c(x) \in L^p(\Omega)$ with $p > n$.* Recall that in this case $s = \min\{r, p\}$. Then we see from (4.7) and (4.8) that

$$\|f\|_{L^s(\Omega)} \le (\Omega, n, p, F, r, s, \varphi, \psi, \|c\|_{L^p(\Omega)}).$$

Arguing as in case (i) above, we obtain the following $W^{4,s}$ estimate of u:

$$\|u\|_{W^{4,s}(\Omega)} \le C(\Omega, n, p, F, r, s, \varphi, \psi, \|c\|_{L^p(\Omega)}).$$

Step 3: Existence of solutions via degree theory. From the $C^{4,\beta}(\overline{\Omega})$ or $W^{4,s}(\Omega)$ estimates for uniformly convex $W^{4,s}(\Omega)$ solutions u of (1.5) in Step 2, we can use the Leray–Schauder degree theory as in [5, 22, 38] to prove the existence of $C^{4,\beta}(\overline{\Omega})$ or $W^{4,s}(\Omega)$ solutions to (1.5) as stated in the theorem. We omit the details. ∎

5. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. As in the proof of Theorem 1.1 in Section 4, we focus on a priori estimates for smooth, uniformly convex solutions. The most crucial ones are Hessian determinant estimates. Without the sign of c, we first need to obtain an a priori L^∞ bound for u.

Lemma 5.1 (A priori L^∞ bound for uniformly convex $W^{4,n}$ solutions). *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be an open, smooth, bounded and uniformly convex domain. Assume that $\varphi \in C^5(\overline{\Omega})$ and $\psi \in C^3(\overline{\Omega})$ with $\min_{\partial\Omega} \psi > 0$. Assume $\mathbf{b} \in L^\infty(\Omega; \mathbb{R}^n)$. Suppose that there exist functions $g_1, g_2 \in L^1(\Omega)$ and a constant $0 \leq m < n-1$ such that*

$$|c(x,z)| \leq |g_1(x)| + |g_2(x)| \cdot |z|^m \quad \text{in } \Omega \times \mathbb{R}. \tag{5.1}$$

Assume that $u \in W^{4,n}(\Omega)$ is a uniformly convex solution to (1.6). Then there exists a constant $C > 0$ depending on Ω, n, φ, ψ, $\|\mathbf{b}\|_{L^\infty(\Omega)}$, $\|g_1\|_{L^1(\Omega)}$, $\|g_2\|_{L^1(\Omega)}$ and m such that

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

Proof. From $u \in W^{4,n}(\Omega)$ and the Sobolev embedding theorem, we have $u \in C^2(\overline{\Omega})$. For a convex function $u \in C^2(\Omega)$ with $u = \varphi$ on $\partial\Omega$, we have (see, e.g., [18, inequality (2.7)])

$$\|u\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\Omega)} + C_1(n, \Omega, \|\varphi\|_{C^2(\Omega)}) \left(\int_{\partial\Omega} (u_\nu^+)^n \, dS \right)^{1/n}, \tag{5.2}$$

where $u_\nu^+ = \max\{0, u_\nu\}$, ν is the unit outer normal of $\partial\Omega$ and dS is the boundary measure. Thus, to prove the lemma, it suffices to prove

$$\int_{\partial\Omega} (u_\nu^+)^n \, dS \leq C(\Omega, n, \varphi, \psi, \|\mathbf{b}\|_{L^\infty(\Omega)}, \|g_1\|_{L^1(\Omega)}, \|g_2\|_{L^1(\Omega)}, m).$$

For this, we use the arguments as in [22, proof of Lemma 4.2]. Observe that since u is convex with boundary value φ on $\partial\Omega$, we have $u_\nu \geq -\|D\varphi\|_{L^\infty(\Omega)}$ and hence

$$|u_\nu| \leq u_\nu^+ + \|D\varphi\|_{L^\infty(\Omega)}, \quad (u_\nu^+)^n \leq u_\nu^n + \|D\varphi\|^n_{L^\infty(\Omega)} \quad \text{on } \partial\Omega. \tag{5.3}$$

Let ρ be a strictly convex defining function of Ω, i.e.

$$\Omega := \{x \in \mathbb{R}^n : \rho(x) < 0\}, \quad \rho = 0 \text{ on } \partial\Omega \quad \text{and} \quad D\rho \neq 0 \text{ on } \partial\Omega.$$

Let

$$\tilde{u} = \varphi + \mu(e^\rho - 1).$$

Then, for μ large, depending on n, Ω and $\|\varphi\|_{C^2(\overline{\Omega})}$, the function $\tilde{u}$ is uniformly convex, and belongs to $C^5(\overline{\Omega})$. Furthermore, as in [18, Lemma 2.1], there exists a constant $C_2 > 0$ depending only on n, Ω, and $\|\varphi\|_{C^4(\overline{\Omega})}$ such that the following facts hold:

(1) $\|\tilde{u}\|_{C^4(\overline{\Omega})} \leq C_2$ and $\det D^2\tilde{u} \geq C_2^{-1} > 0$ in Ω,

(2) letting $\tilde{w} = [\det D^2\tilde{u}]^{-1}$, and denoting by $(\tilde{U}^{ij})$ the cofactor matrix of $D^2\tilde{u}$, we have

$$\|\tilde{U}^{ij} D_{ij}\tilde{w}\|_{L^\infty(\Omega)} \leq C_2.$$

Let $K(x)$ be the Gauss curvature at $x \in \partial\Omega$. Since Ω is uniformly convex, we have

$$0 < C^{-1}(\Omega) \leq K(x) \leq C(\Omega) \quad \text{on } \partial\Omega. \tag{5.4}$$

From [22, (4.10) in the proof of Lemma 4.2] with $\theta = 0$ and $f_\delta := -\Delta u + \mathbf{b}\cdot Du + c$ which uses (i) and (ii) above , we obtain

$$\int_{\partial\Omega} K\psi u_\nu^n \, dS \le \int_\Omega (\Delta u - \mathbf{b}\cdot Du - c)(u-\tilde{u})\, dx + C_3\left(\int_{\partial\Omega} (u_\nu^+)^n \, dS\right)^{(n-1)/n} + C_3, \tag{5.5}$$

where C_3 depends on C_2, Ω and φ.

We will estimate the first term on the right-hand side of (5.5) by splitting it into three terms. Firstly, using $u\Delta u = \operatorname{div}(uDu) - |Du|^2$ and integrating by parts, we have

$$\begin{aligned}\int_\Omega \Delta u(u-\tilde{u})\, dx &\le \int_\Omega u\Delta u\, dx + C_2\int_\Omega \Delta u\, dx \\ &= \int_{\partial\Omega} \varphi u_\nu \, dS - \int_\Omega |Du|^2\, dx + C_2 \int_{\partial\Omega} u_\nu \, dS \\ &\le C(\varphi, C_2)\int_{\partial\Omega} |u_\nu|\, dS - \int_\Omega |Du|^2\, dx \\ &\le C_4(n,\varphi,C_2)\left(\int_{\partial\Omega}(u_\nu^+)^n\, dS\right)^{1/n} + C_4(n,\varphi,C_2) \ \ \text{(recalling (5.3))}.\end{aligned} \tag{5.6}$$

Secondly, by integration by parts, we find

$$\begin{aligned}\int_\Omega |Du|^2\, dx &= \int_\Omega (\operatorname{div}(uDu) - u\Delta u)\, dx = \int_{\partial\Omega}\varphi u_\nu\, dS - \int_\Omega u\Delta u\, dx \\ &\le C_5(\varphi)\int_{\partial\Omega} u_\nu^+\, dS + \|u\|_{L^\infty(\Omega)}\int_\Omega \Delta u\, dx + C_5(\varphi) \\ &\le (C_5 + \|u\|_{L^\infty(\Omega)})\int_{\partial\Omega} u_\nu^+\, dS + C_5.\end{aligned} \tag{5.7}$$

In view of (5.7) with (5.2), we can estimate

$$\begin{aligned}\int_\Omega \mathbf{b}\cdot Du(\tilde{u}-u)\, dx &\le |\Omega|^{1/2}\|\mathbf{b}\|_{L^\infty(\Omega)}(\|\tilde{u}\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)})\left(\int_\Omega |Du|^2\, dx\right)^{1/2} \\ &\le C_6 + C_6\left(\int_{\partial\Omega}(u_\nu^+)^n\, dS\right)^{2/n},\end{aligned} \tag{5.8}$$

where C_6 depends on Ω, n, φ and $\|\mathbf{b}\|_{L^\infty(\Omega)}$, the dependence on $\|\mathbf{b}\|_{L^\infty(\Omega)}$ being linear.

Finally, using (5.1) and (5.2), we have

$$\begin{aligned}\int_\Omega -c(u-\tilde{u})\, dx &\le (\|u\|_{L^\infty(\Omega)} + \|\tilde{u}\|_{L^\infty(\Omega)})\int_\Omega (|g_1| + |g_2|\,|u|^m)\, dx \\ &\le C + C\|u\|^{m+1}_{L^\infty(\Omega)} \\ &\le C_7 + C_7\left(\int_{\partial\Omega}(u_\nu^+)^n\, dS\right)^{\frac{m+1}{n}}.\end{aligned} \tag{5.9}$$

Here C_7 depends on Ω, n, φ, $\|g_1\|_{L^1(\Omega)}$, $\|g_2\|_{L^1(\Omega)}$ and m.

It follows from (5.3) that

$$\int_{\partial\Omega} K\psi(u_\nu^+)^n\,dS \le C_8(\Omega,\varphi,\psi) + \int_{\partial\Omega} K\psi u_\nu^n\,dS. \tag{5.10}$$

Combining (5.4)–(5.6) and (5.8)–(5.10) while recalling that $0 \le m < n-1$ and $n \ge 3$, we obtain

$$\begin{aligned} C^{-1}(\Omega)\min_{\partial\Omega}\psi \int_{\partial\Omega}(u_\nu^+)^n\,dS &\le C_8 + \int_{\partial\Omega} K\psi u_\nu^n\,dS \\ &\le C_9\Big[1 + \Big(\int_{\partial\Omega}(u_\nu^+)^n\,dS\Big)^{\frac{n-1}{n}} + \Big(\int_{\partial\Omega}(u_\nu^+)^n\,dS\Big)^{\frac{m+1}{n}}\Big], \end{aligned}$$

where C_9 depends on C_3, C_4, C_6, C_7 and C_8. It follows that

$$\int_{\partial\Omega}(u_\nu^+)^n\,dS \le C,$$

where C depends on Ω, n, φ, ψ, $\|\mathbf{b}\|_{L^\infty(\Omega)}$, $\|g_1\|_{L^1(\Omega)}$, $\|g_2\|_{L^1(\Omega)}$ and m. The proof of the lemma is complete. ∎

Remark 5.2. We have the following observations regarding the two-dimensional version of Lemma 5.1.

(i) The above proof fails in two dimensions, because the right-hand side of (5.8) is then of the same order of magnitude as the left-hand side of (5.5). Therefore, when C_6 is large, plugging (5.8) into (5.5) does not give any new information.

(ii) On the other hand, since C_6 depends linearly on $\|\mathbf{b}\|_{L^\infty(\Omega)}$, in two dimensions one can still absorb the right-hand side of (5.8) into the left-hand side of (5.5) as long as $\|\mathbf{b}\|_{L^\infty(\Omega)}$ is small, depending on Ω, φ and ψ. In this case, we still have an L^∞ estimate.

(iii) In Section 6, we will establish an L^∞ estimate in two dimensions under a stronger condition on $\mathbf{b}$ but allowing $\|\mathbf{b}\|_{L^\infty(\Omega)}$ to be arbitrarily large.

Next, we establish the Hessian determinant estimates.

Lemma 5.3 (Hessian determinant estimates). *Let $u \in W^{4,p}(\Omega)$ be a uniformly convex solution to the fourth order equation*

$$\begin{cases} \sum_{i,j=1}^n U^{ij}D_{ij}w = -\Delta u + \mathbf{b}\cdot Du + c(x) & \text{in } \Omega, \\ w = (\det D^2u)^{-1} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ w = \psi & \text{on } \partial\Omega, \end{cases} \tag{5.11}$$

where $(U^{ij}) = (\det D^2u)(D^2u)^{-1}$, $\min_{\partial\Omega}\psi > 0$, $\mathbf{b} \in L^\infty(\Omega;\mathbb{R}^n)$ and $c \in L^p(\Omega)$ with

$p > 2n$. Then there exists a constant $C > 0$ depending on Ω, n, p, φ, ψ, $\|\mathbf{b}\|_{L^\infty(\Omega)}$ and $\|c\|_{L^p(\Omega)}$ such that

$$0 < C^{-1} \leq \det D^2 u \leq C \quad \textit{in}\ \Omega.$$

Proof. The proof uses a trick in Chau–Weinkove [5]. For simplicity, denote

$$d := \det D^2 u \quad \text{and} \quad (u^{ij}) = (D^2 u)^{-1}.$$

Let

$$G := d e^{Mu^2},$$

where $M > 0$ is a large constant to be determined. By Lemma 5.1, we have

$$\|u\|_{L^\infty(\Omega)} \leq C_0,$$

where $C_0 > 0$ depends on Ω, n, φ, ψ, $\|\mathbf{b}\|_{L^\infty(\Omega)}$ and $\|c\|_{L^1(\Omega)}$.

Since $w = d^{-1}$, we have $w = G^{-1}e^{Mu^2}$. Direct calculations yield

$$\begin{aligned}
D_i w &= -G^{-2} D_i G e^{Mu^2} + 2MuD_i u G^{-1} e^{Mu^2},\\
D_{ij} w &= 2G^{-3} D_i G D_j G e^{Mu^2} - G^{-2} D_{ij} G e^{Mu^2}\\
&\quad - 2MuD_j u D_i G G^{-2} e^{Mu^2} - 2MuD_i u D_j G G^{-2} e^{Mu^2}\\
&\quad + 2MD_i u D_j u G^{-1} e^{Mu^2} + 2MuD_{ij} u G^{-1} e^{Mu^2} + 4M^2 u^2 D_i u D_j u G^{-1} e^{Mu^2}.
\end{aligned}$$

Then, using $U^{ij} G^{-1} e^{Mu^2} = u^{ij}$, we have

$$\begin{aligned}
U^{ij} D_{ij} w &= 2G^{-2} u^{ij} D_i G D_j G - G^{-1} u^{ij} D_{ij} G - 4MuG^{-1} u^{ij} D_i u D_j G\\
&\quad + 2Mu^{ij} D_i u D_j u + 2Mnu + 4M^2 u^2 u^{ij} D_i u D_j u\\
&= G^{-2} u^{ij} D_i G D_j G + u^{ij}(2MuD_i u - G^{-1} D_i G)(2MuD_j u - G^{-1} D_j G)\\
&\quad - G^{-1} u^{ij} D_{ij} G + 2Mu^{ij} D_i u D_j u + 2Mnu\\
&\geq -G^{-1} u^{ij} D_{ij} G + 2Mu^{ij} D_i u D_j u + 2Mnu.
\end{aligned}$$

Thus, from the first equation in (5.11), we obtain

$$G^{-1} u^{ij} D_{ij} G \geq 2Mu^{ij} D_i u D_j u + 2Mnu + \Delta u - \mathbf{b} \cdot Du - c.$$

Using the matrix inequality (see, for example, [20, Lemma 2.8 (c)])

$$u^{ij} D_i v D_j v \geq \frac{|Dv|^2}{\operatorname{trace}(D^2 u)} = \frac{|Dv|^2}{\Delta u}$$

together with $\Delta u \geq n d^{1/n}$, we find that

$$\begin{aligned}
G^{-1} u^{ij} D_{ij} G &\geq 2M \frac{|Du|^2}{\Delta u} + \frac{1}{2}\Delta u - \mathbf{b} \cdot Du + \frac{1}{2}\Delta u + 2Mnu - c\\
&\geq 2\sqrt{M}|Du| - |\mathbf{b}| \cdot |Du| + (n/2) d^{1/n} + 2Mnu - c\\
&\geq -(c - 2Mnu - (n/2) d^{1/n})^+ \quad \text{in } \Omega \qquad (5.12)
\end{aligned}$$

provided
$$M \geq \tfrac{1}{4}\|\mathbf{b}\|^2_{L^\infty(\Omega)}.$$

Hence, by the ABP estimate applied to (5.12) in Ω where $G = \psi^{-1}e^{M\varphi^2}$ on $\partial\Omega$, we have

$$\begin{aligned}
\sup_\Omega G &\leq \sup_{\partial\Omega}(\psi^{-1}e^{M\varphi^2}) + C(n,\Omega)\left\|\frac{(c - 2Mnu - (n/2)d^{1/n})^+}{[\det(G^{-1}(D^2u)^{-1})]^{1/n}}\right\|_{L^n(\Omega)}\\
&= \sup_{\partial\Omega}(\psi^{-1}e^{M\varphi^2}) + C(n,\Omega)\left\|\frac{de^{Mu^2}(c - 2Mnu - (n/2)d^{1/n})^+}{d^{-1/n}}\right\|_{L^n(\Omega)}\\
&\leq \sup_{\partial\Omega}(\psi^{-1}e^{M\varphi^2}) + C_1\|d^{1+1/n}(c - 2Mnu - (n/2)d^{1/n})^+\|_{L^n(\Omega)}. \qquad (5.13)
\end{aligned}$$

Here C_1 depends on n, Ω and C_0 (via $\|u\|_{L^\infty(\Omega)}$). Note that, for any $p_0 > n$, we have

$$\begin{aligned}
&\|d^{1+1/n}(c - 2Mnu - (n/2)d^{1/n})^+\|_{L^n(\Omega)}\\
&\quad\leq \left(\int_{\{c-2Mnu\geq(n/2)d^{1/n}\}} d^{n+1}(c - 2Mnu)^n\,dx\right)^{1/n}\\
&\quad\leq \left(\int_{\{c-2Mnu\geq(n/2)d^{1/n}\}} d^{n+1}(c - 2Mnu)^n\frac{(c - 2Mnu)^{p_0-n}}{[(n/2)d^{1/n}]^{p_0-n}}\,dx\right)^{1/n}\\
&\quad= (n/2)^{-\frac{p_0-n}{n}}\left(\int_{\{c-2Mnu\geq(n/2)d^{1/n}\}} d^{n-p_0/n+2}(c - 2Mnu)^{p_0}\,dx\right)^{1/n}. \qquad (5.14)
\end{aligned}$$

We now choose p_0 such that
$$2n < p_0 < \min\{n(n+2), p\}.$$

Let $\gamma = 1 - \frac{p_0}{n^2} + \frac{2}{n}$. Then $0 < \gamma < 1$. Moreover, from (5.13) and (5.14), we have

$$\begin{aligned}
\sup_\Omega G &\leq C + C\left(\int_\Omega d^{n\gamma}(|c| + |u|)^{p_0}\,dx\right)^{1/n}\\
&\leq C + C\left(\int_\Omega (de^{Mu^2})^{n\gamma}(|c|^p + 1)\,dx\right)^{1/n}\\
&\leq C_2 + C_2\Big(\sup_\Omega G\Big)^\gamma\cdot\left(\int_\Omega(|c|^p + 1)\,dx\right)^{1/n}.
\end{aligned}$$

Here C_2 depends on Ω, M, φ, ψ, C_1, γ and p. It follows that
$$\sup_\Omega G \leq C_3(C_2, \gamma, \|c\|_{L^p(\Omega)}).$$

Since $G = de^{Mu^2}$, we also get an upper bound for $d = \det D^2u$:
$$\det D^2u \leq C_3 \quad \text{in } \Omega.$$

It remains to establish a positive lower bound for $\det D^2u$.

Once we have the upper bound of the Hessian determinant of u, using $u = \varphi$ on $\partial\Omega$ and a suitable barrier, we obtain

$$\sup_\Omega |u| + \sup_\Omega |Du| \le C_4(C_3, \varphi, \Omega).$$

Then we can apply the Legendre transform to get a lower bound of the determinant. According to Proposition 2.4, the Legendre transform u^* of u satisfies

$$u^{*ij} D_{ij}(w^* + |y|^2/2) = \mathbf{b}(Du^*) \cdot y + c(Du^*) \quad \text{in } \Omega^* := Du(\Omega),$$

where $(u^{*ij}) := (D^2u^*)^{-1}$ and $w^* := \log\det D^2u^*$. Applying the ABP estimate to $w^* + |y|^2/2$ on Ω^*, and then changing variables $y = Du(x)$ with $dy = \det D^2u\, dx$, we obtain

$$\begin{aligned}
&\sup_{\Omega^*}(w^* + |y|^2/2) \\
&\quad\le \sup_{\partial\Omega^*}(w^* + |y|^2/2) + C(n)\operatorname{diam}(\Omega^*)\left\| \frac{\mathbf{b}(Du^*)\cdot y + c(Du^*)}{(\det u^{*ij})^{1/n}}\right\|_{L^n(\Omega^*)} \\
&\quad\le C(\psi, C_4) + C(n, C_4)\left(\int_{\Omega^*} \frac{|\mathbf{b}(Du^*)\cdot y + c(Du^*)|^n}{(\det D^2u^*)^{-1}}\, dy\right)^{1/n} \\
&\quad= C(\psi, C_4) + C(n, C_4)\left(\int_\Omega |\mathbf{b}\cdot Du + c(x)|^n\, dy\right)^{1/n} \\
&\quad\le C(\psi, C_4) + C(n, C_4)\Big(\|\mathbf{b}\|_{L^n(\Omega)}\sup_\Omega |Du| + \|c\|_{L^n(\Omega)}\Big).
\end{aligned}$$

In particular, we have

$$\sup_{\Omega^*} w^* \le C_5,$$

where $C_5 > 0$ depends on $\Omega, n, \varphi, \psi, \|\mathbf{b}\|_{L^\infty(\Omega)}$ and $\|c\|_{L^p(\Omega)}$. Since $w^* = \log\det D^2u^*$, the above estimate gives a lower bound for $\det D^2u$:

$$\det D^2u \ge e^{-C_5} \quad \text{in } \Omega,$$

completing the proof of the lemma. ∎

Remark 5.4. If there is no first order term $\mathbf{b}\cdot Du$ on the right-hand of (1.6), we can directly obtain Hessian determinant bounds by the same trick as in the proof of Lemma 5.3 without getting an a priori L^∞ bound of u. Moreover, these bounds are valid for all dimensions.

Proof of Theorem 1.2. The proof uses a priori estimates and degree theory as in the proof of Theorem 1.1. We obtain the existence of a uniformly convex solution in $C^{4,\alpha}(\overline{\Omega})$ in case (i), and in $W^{4,p}(\Omega)$ in case (ii), with the stated estimates provided that we can establish the latter estimates for $W^{4,p}(\Omega)$ solutions. Thus, it remains to establish these a priori estimates.

Assume now $u \in W^{4,p}(\Omega)$ is a uniformly convex smooth solution to (1.6). By Lemma 5.1 and the assumption on c in either (i) or (ii), we can obtain Hessian determinant estimates for u by Lemma 5.3. Once we have those estimates, Theorem 3.4 applies with

$$F(x) = |x|^2/2 \quad \text{and} \quad g(x) = \mathbf{b}(x) \cdot Du(x) + c(x).$$

This gives Hölder estimates for w. The rest of the proof of Theorem 1.2, which is concerned with global higher order derivative estimates, is similar to Step 2 in the proof of Theorem 1.1 (i, ii). We omit the details. ∎

Remark 5.5. In two dimensions, when $\|\mathbf{b}\|_{L^\infty(\Omega)}$ is small, depending on Ω, ψ and ψ, the conclusions of Theorem 1.2 still hold. Indeed, in this case, by Remark 5.2, we still have the L^∞ estimate of Lemma 5.1. The conclusion of Theorem 1.2 then follows.

6. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3. As in the proof of Theorem 1.1, it suffices to derive a priori estimates for $W^{4,p}(\Omega)$ solutions. Here, we recall that

$$p > 2.$$

Theorem 1.3 can be deduced from the following result.

Theorem 6.1 (A priori $W^{4,p}(\Omega)$ estimates for $W^{4,p}(\Omega)$ solutions). *Let $\Omega \subset \mathbb{R}^2$, φ, ψ, $\mathbf{b}$ and c be as in Theorem 1.3. Assume that $u \in W^{4,p}(\Omega)$ is a uniformly convex solution to (1.6). Then*

$$\|u\|_{W^{4,p}(\Omega)} \le C,$$

where $C > 0$ is a constant depending on Ω, p, φ, ψ, $\mathbf{b}$ and c.

The rest of this section is devoted to the proof of Theorem 6.1.

We will first obtain an L^∞ bound of u and an L^2 bound of Du. For this, the following Poincaré type inequality will be useful.

Lemma 6.2 (Poincaré type inequality on planar convex domains). *Let $\Omega \subset \mathbb{R}^2$ be an open, smooth, bounded and uniformly convex domain. Assume that $u \in C^1(\Omega) \cap C(\overline{\Omega})$ and $u|_{\partial\Omega} = \varphi$. Then*

$$\int_\Omega |u|^2\, dx \le C(\varphi, \operatorname{diam}(\Omega))\|u\|_{L^\infty(\Omega)} + \frac{\operatorname{diam}(\Omega)^2}{16} \int_\Omega |Du|^2\, dx.$$

Proof. Note that for any one-variable function $f \in C^1(a,b) \cap C^0[a,b]$ where $a < b$, one has

$$\int_a^b |f(x)|^2\, dx \le (b-a)(|f(a)| + |f(b)|)\|f\|_{L^\infty(a,b)} + \frac{(b-a)^2}{8} \int_a^b |f'(x)|^2\, dx. \tag{6.1}$$

Indeed, denoting $c := \frac{a+b}{2}$, then using Hölder's inequality and Fubini's theorem, one obtains

$$\begin{aligned}
\int_a^c |f(x)|^2\,dx &= \int_a^c f(a)(2f(x) - f(a))\,dx + \int_a^c \Big(\int_a^x f'(t)\,dt\Big)^2\,dx \\
&\leq 2(c-a)|f(a)|\cdot\|f\|_{L^\infty(a,b)} - (c-a)f(a)^2 + \int_a^c (x-a)\int_a^x |f'(t)|^2\,dt\,dx \\
&= 2(c-a)|f(a)|\cdot\|f\|_{L^\infty(a,b)} - (c-a)f(a)^2 + \int_a^c |f'(t)|^2\int_t^c (x-a)\,dx\,dt \\
&\leq 2(c-a)|f(a)|\cdot\|f\|_{L^\infty(a,b)} - (c-a)f(a)^2 + \frac{(c-a)^2}{2}\int_a^c |f'(x)|^2\,dx \\
&\leq (b-a)|f(a)|\cdot\|f\|_{L^\infty(a,b)} + \frac{(b-a)^2}{8}\int_a^c |f'(x)|^2\,dx. \qquad (6.2)
\end{aligned}$$

Similarly, we have

$$\int_c^b |f(x)|^2\,dx \leq (b-a)|f(b)|\cdot\|f\|_{L^\infty(a,b)} + \frac{(b-a)^2}{8}\int_c^b |f'(x)|^2\,dx. \qquad (6.3)$$

Combining (6.2) with (6.3), we obtain (6.1).

Next, by the convexity of Ω, we can assume that there are $c, d \in \mathbb{R}$ and one-variable functions $a(x_1)$, $b(x_1)$ such that

$$\Omega = \{(x_1, x_2) : c < x_1 < d,\ a(x_1) < x_2 < b(x_1)\}.$$

It is clear that $d - c \leq \operatorname{diam}(\Omega)$ and $b(x_1) - a(x_1) \leq \operatorname{diam}(\Omega)$. Then, by (6.1) and $u = \varphi$ on $\partial\Omega$, we have

$$\begin{aligned}
\int_{a(x_1)}^{b(x_1)} |u(x_1, x_2)|^2\,dx_2 &\leq 2\operatorname{diam}(\Omega)\|\varphi\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)} \\
&\quad + \frac{\operatorname{diam}(\Omega)^2}{8}\int_{a(x_1)}^{b(x_1)} |D_{x_2}u(x_1, x_2)|^2\,dx_2.
\end{aligned}$$

Integrating the above inequality over $c < x_1 < d$ yields

$$\int_\Omega |u|^2\,dx \leq 2\operatorname{diam}(\Omega)^2\|\varphi\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)} + \frac{\operatorname{diam}(\Omega)^2}{8}\int_\Omega |D_{x_2}u|^2\,dx. \qquad (6.4)$$

Similarly,

$$\int_\Omega |u|^2\,dx \leq 2\operatorname{diam}(\Omega)^2\|\varphi\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)} + \frac{\operatorname{diam}(\Omega)^2}{8}\int_\Omega |D_{x_1}u|^2\,dx. \qquad (6.5)$$

Combining (6.4) and (6.5), we obtain

$$\int_\Omega |u|^2\,dx \leq 2\operatorname{diam}(\Omega)^2\|\varphi\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)} + \frac{\operatorname{diam}(\Omega)^2}{16}\int_\Omega |Du|^2\,dx,$$

completing the proof of the lemma. ∎

6.1. Estimates for $\sup_\Omega |u|$ *and* $\|Du\|_{L^2(\Omega)}$

Now we derive bounds for u and $\|Du\|_{L^2(\Omega)}$.

Lemma 6.3 (L^∞ and $W^{1,2}$ estimates). *Let* $\Omega \subset \mathbb{R}^2$, φ, ψ, $\mathbf{b}$ *and* c *be as in Theorem 1.3. Assume that* $u \in W^{4,p}(\Omega)$ *is a uniformly convex solution to* (1.6). *Then there exists a constant* $C > 0$ *depending on* Ω, φ, ψ, $\mathbf{b}$ *and* $\|c\|_{L^1(\Omega)}$ *such that*

$$\|u\|_{L^\infty(\Omega)} \leq C \quad \text{and} \quad \|Du\|_{L^2(\Omega)} \leq C.$$

Proof. To prove the lemma where $n = 2$, by (5.2) and (5.7) it suffices to prove

$$\int_{\partial\Omega} u_\nu^2 \, dS \leq C(\Omega, \varphi, \psi, \mathbf{b}, \|c\|_{L^1(\Omega)}), \tag{6.6}$$

where ν is the unit outer normal to $\partial\Omega$.

Let $\tilde{u}$ be as in the proof of Lemma 5.1 so that (i) and (ii) there are satisfied. Let $K(x)$ be the Gauss curvature at $x \in \partial\Omega$. Then, as in (5.5), we have, for some $C_1(\Omega, \varphi) > 0$,

$$\int_{\partial\Omega} K\psi u_\nu^2 \, dS \leq \int_\Omega (\Delta u - \mathbf{b} \cdot Du - c)(u - \tilde{u}) \, dx + C_1 \left(\int_{\partial\Omega} u_\nu^2 \, dS \right)^{1/2} + C_1. \tag{6.7}$$

Next, we will estimate the RHS of (6.7) term by term. First, from the inequality before last in (5.6), we have

$$\int_\Omega \Delta u(u - \tilde{u}) \, dx \leq C(\Omega, \varphi) \left(\int_{\partial\Omega} u_\nu^2 \, dS \right)^{1/2} - \int_\Omega |Du|^2 \, dx. \tag{6.8}$$

Using $u = \varphi$ on $\partial\Omega$, and integrating by parts, we get

$$\begin{aligned}
\int_\Omega (\mathbf{b} \cdot Du)\tilde{u} \, dx &= \int_\Omega (\mathbf{b}\tilde{u}) \cdot Du \, dx \\
&= \int_{\partial\Omega} u\tilde{u}\mathbf{b} \cdot \nu \, dS - \int_\Omega \operatorname{div}(\mathbf{b}\tilde{u}) u \, dx \\
&= \int_{\partial\Omega} \varphi\tilde{u}(\mathbf{b} \cdot \nu) \, dS - \int_\Omega (\mathbf{b} \cdot D\tilde{u} + \tilde{u} \operatorname{div} \mathbf{b}) u \, dx \\
&\leq C_2(1 + \|u\|_{L^\infty(\Omega)}) \leq C_3 + C_3 \left(\int_{\partial\Omega} u_\nu^2 \, dS \right)^{1/2},
\end{aligned} \tag{6.9}$$

where C_3 depends on Ω, φ, $\sup_{\partial\Omega} |\mathbf{b}|$, $\|\mathbf{b}\|_{L^\infty(\Omega)}$ and $\|\operatorname{div} \mathbf{b}\|_{L^\infty(\Omega)}$.

Moreover,

$$\int_\Omega -(\mathbf{b} \cdot Du) u \, dx = \frac{1}{2} \int_\Omega -\mathbf{b} \cdot D(u^2) \, dx = \frac{1}{2} \left[\int_\Omega (\operatorname{div} \mathbf{b}) u^2 \, dx - \int_{\partial\Omega} u^2 \mathbf{b} \cdot \nu \, dS \right].$$

Note that $\operatorname{div} \mathbf{b} \leq 32/\operatorname{diam}(\Omega)^2$. Then by Lemma 6.2 and (5.2), we have

$$\begin{aligned}
\frac{1}{2} \int_\Omega (\operatorname{div} \mathbf{b}) u^2 \, dx &\leq C(\varphi, \operatorname{diam}(\Omega)) \|u\|_{L^\infty(\Omega)} + \int_\Omega |Du|^2 \, dx \\
&\leq C(\Omega, \varphi) + C(\Omega, \varphi) \left(\int_{\partial\Omega} u_\nu^2 \, dS \right)^{1/2} + \int_\Omega |Du|^2 \, dx.
\end{aligned}$$

Hence

$$\int_\Omega -(\mathbf{b}\cdot Du)u\,dx = \frac{1}{2}\Big[\int_\Omega (\operatorname{div}\mathbf{b})u^2\,dx - \int_{\partial\Omega}\varphi^2\mathbf{b}\cdot\nu\,dS\Big]$$
$$\leq C_4 + C_4\Big(\int_{\partial\Omega} u_\nu^2\,dS\Big)^{1/2} + \int_\Omega |Du|^2\,dx, \tag{6.10}$$

where C_4 depends on Ω, φ and $\sup_{\partial\Omega}|\mathbf{b}|$.

Finally, as in (5.9), we get

$$\int_\Omega -c(u-\tilde{u})\,dx \leq C_5 + C_5\Big(\int_{\partial\Omega} u_\nu^2\,dS\Big)^{1/2}, \tag{6.11}$$

where C_5 depends on Ω, φ and $\|c\|_{L^1(\Omega)}$.

Combining (6.7)–(6.11), we obtain

$$C^{-1}(\Omega)\inf_{\partial\Omega}\psi\int_{\partial\Omega} u_\nu^2\,dS \leq \int_{\partial\Omega} K\psi u_\nu^2\,dS \leq C_6\Big[1+\Big(\int_{\partial\Omega} u_\nu^2\,dS\Big)^{1/2}\Big],$$

where $C_6 > 0$ depends on Ω, φ, ψ, $\mathbf{b}$ and $\|c\|_{L^1(\Omega)}$. From this, we deduce (6.6), completing the proof of the lemma. ■

6.2. *Hessian determinant estimates for u*

Lemma 6.4 (Hessian determinant estimates). *Let $\Omega\subset\mathbb{R}^2$, φ, ψ, $\mathbf{b}$ and c be as in Theorem 1.3. Assume that $u\in W^{4,p}(\Omega)$ is a uniformly convex solution to (1.6). Then*

$$0 < C^{-1} \leq \det D^2u \leq C \quad \text{in } \Omega,$$

where $C > 0$ is a constant depending on Ω, φ, ψ, $\mathbf{b}$ and $\|c\|_{L^2(\Omega)}$.

Proof. We first prove the lower bound of $\det D^2u$. Note that in two dimensions, we have trace $U = \Delta u$. Hence we can rewrite the first equation in (1.6) as

$$U^{ij}D_{ij}(w + |x|^2/2) = \mathbf{b}(x)\cdot Du(x) + c(x) =: Q(x) \quad \text{in } \Omega. \tag{6.12}$$

By Lemma 6.3, we have

$$\|Q\|_{L^2(\Omega)} \leq C_0,$$

where C_0 depends on Ω, φ, ψ, $\mathbf{b}$ and $\|c\|_{L^2(\Omega)}$.

Applying the ABP estimate to (6.12) and using $\det U = \det D^2u$, we have

$$\sup_\Omega(w+|x|^2/2) \leq \sup_{\partial\Omega}\psi + C(\Omega) + C(\Omega)\Big\|\frac{Q}{(\det U)^{1/2}}\Big\|_{L^2(\Omega)}$$
$$\leq C(\Omega,\psi) + C(\Omega)\|Q\|_{L^2(\Omega)}\cdot\sup_\Omega(\det D^2u)^{-1/2}$$
$$\leq C(\Omega,\psi) + C(\Omega)\Big(\sup_\Omega w\Big)^{1/2}.$$

Therefore $\sup_\Omega w \le C_1$, where C_1 depends on Ω, φ, ψ, $\mathbf{b}$ and $\|c\|_{L^2(\Omega)}$. Consequently,

$$\det D^2 u \ge C_1^{-1} > 0 \quad \text{in } \Omega. \tag{6.13}$$

Hence by the boundary Hölder estimate for solutions of nonuniformly elliptic equations [17, Proposition 2.1], we know from (6.12) that w is Hölder continuous on $\partial\Omega$ with estimates depending only on C_1, Ω and ψ. Then by constructing a suitable barrier near the boundary as in [18, Lemma 2.5], we can obtain

$$\|Du\|_{L^\infty(\Omega)} \le C_2,$$

where C_2 depends on C_1, Ω, φ and ψ.

The upper bound of the Hessian determinant can be obtained similar to Lemma 5.3. Let $u^*(y)$ be the Legendre transform of $u(x)$ where

$$y = Du(x) \in Du(\Omega) =: \Omega^*.$$

Then

$$\operatorname{diam}(\Omega^*) \le C_2.$$

By Proposition 2.4 (with $F(x) = |x|^2/2$), u^* satisfies

$$U^{*ij} D_{ij}(-w^* - |y|^2/2) = -Q(Du^*) \det D^2 u^* \quad \text{in } \Omega^*, \tag{6.14}$$

where $(U^{*ij}) := (\det D^2 u^*)(D^2 u^*)^{-1}$ and $w^* = \log\det D^2 u^*$.

Applying the ABP maximum principle to (6.14), and recalling that

$$w^*(y) = \log(\det D^2 u(x))^{-1} = \log w(x) = \log \psi(x) \quad \text{on } \partial\Omega^*,$$

we obtain

$$\begin{aligned} &\sup_{\Omega^*}(-w^* - |y|^2/2) \\ &\qquad \le \sup_{\partial\Omega^*}(-w^* - |y|^2/2) + C(\operatorname{diam}(\Omega^*))\|Q(Du^*)(\det D^2 u^*)^{1/2}\|_{L^2(\Omega^*)} \\ &\qquad \le -\log \min_{\partial\Omega} \psi + C(C_2)\|Q\|_{L^2(\Omega)}, \end{aligned}$$

where we have used

$$\begin{aligned} \int_{\Omega^*} [Q(Du^*)]^2 \det D^2 u^* \, dy &= \int_\Omega [Q(x)]^2 \det D^2 u^* \det D^2 u \, dx \\ &= \int_\Omega [Q(x)]^2 \, dx = \|Q\|^2_{L^2(\Omega)}. \end{aligned}$$

Therefore,

$$\sup_{\Omega^*}(-w^*) \le C_3,$$

where C_3 depends on C_0, C_2 and $\min_{\partial\Omega}\psi$. This implies $w^* \ge -C_3$ in Ω^*, and hence

$$\det D^2 u \le e^{C_3} \quad \text{in } \Omega. \tag{6.15}$$

The lemma follows from (6.13) and (6.15). ∎

6.3. Proof of Theorem 6.1

Finally, we can prove Theorem 6.1 which implies Theorem 1.3.

Proof of Theorem 6.1. Once we have the determinant estimates, we can establish higher estimates by using the regularity of the linearized Monge–Ampère equation with drift terms as in Sections 4 and 5. In two dimensions, we can also establish these estimates as in [22].

By Lemma 6.4, we have

$$0 < \lambda \leq \det D^2 u \leq \Lambda \quad \text{in } \Omega \tag{6.16}$$

for λ, Λ depending on Ω, φ, ψ, $\mathbf{b}$, p and $\|c\|_{L^p(\Omega)}$. By the interior $W^{2,1+\varepsilon}$ estimates for Monge–Ampère equations [8, 13, 35], we have $D^2 u \in L^{1+\varepsilon}_{\mathrm{loc}}(\Omega)$ for some constant $\varepsilon(\lambda, \Lambda) > 0$. By the global $W^{2,1+\varepsilon}$ estimates for Monge–Ampère equations [33], there exists a constant $C_0 > 0$ depending on Ω, φ, ψ, $\mathbf{b}$, p and $\|c\|_{L^p(\Omega)}$ such that

$$\|u\|_{W^{2,1+\varepsilon(\lambda,\Lambda)}(\Omega)} \leq C_0.$$

Let $q := \min\{p, 1 + \varepsilon(\lambda, \Lambda)\} > 1$. Then

$$G := -\Delta u + \mathbf{b} \cdot Du + c$$

satisfies

$$\|G\|_{L^q(\Omega)} \leq C_1,$$

where $C_1 > 0$ depends on Ω, p, φ, ψ, $\mathbf{b}$ and $\|c\|_{L^p(\Omega)}$. Recall that

$$U^{ij} D_{ij} w = G \text{ on } \Omega, \quad w = \psi \text{ on } \partial\Omega.$$

By the global Hölder estimate for linearized Monge–Ampère equations [26] with L^q right-hand side where $q > n/2$, we deduce

$$\|w\|_{C^\alpha(\bar{\Omega})} \leq C(\Omega, \varphi, \psi, p, \mathbf{b}, c),$$

where $\alpha \in (0, 1)$ depends on $\Omega, \varphi, \psi, p, \mathbf{b}, c$. The proof of the $W^{4,p}(\Omega)$ estimate for u is now the same as that of Theorem 1.1 (ii). Hence, the theorem is proved. ■

7. Extensions and the proof of Theorem 3.2

In this section, we discuss (1.1) with more general lower order terms, and present a proof of Theorem 3.2 for completeness.

7.1. Possible extensions of the main results

The following remarks indicate some possible extensions of our main results.

Remark 7.1. From the proofs in Sections 4–6 and the L^∞ estimates in Lemma 5.1, it can be seen that some conclusions of Theorems 1.1–1.3 also hold for more general cases of $c = c(x, z)$. Consider, for example,

$$c(x, z) = g_1(x) + g_2(x)h(z).$$

Then the following facts hold:

(1) The conclusions in Theorem 1.1 (ii) hold when $g_1 \le 0$, $g_2 \le 0$, $g_1, g_2 \in L^p(\Omega)$ with $p > n$, and $h \ge 0$ with $h \in C^\alpha(\mathbb{R})$.

(2) The conclusions in Theorem 1.2 (i) hold when $g_1, g_2 \in C^\alpha(\overline{\Omega})$ and $h \in C^\alpha(\mathbb{R})$ with $|h(z)| \le C|z|^m$ for $0 \le m < n - 1$.

(3) The conclusions in Theorem 1.2 (ii) hold when $g_1, g_2 \in L^p(\Omega)$ with $p > 2n$, and $h \in C^\alpha(\mathbb{R})$ with $|h(z)| \le C|z|^m$ for $0 \le m < n - 1$.

(4) The conclusions in Theorem 1.3 hold when $g_1, g_2 \in L^p(\Omega)$ with $p > 2$, and $h \in C^\alpha(\mathbb{R})$ with $|h(z)| \le C|z|^m$ for $0 \le m < 1$.

Remark 7.2. Since we use the trace of $\mathbf{b}$ on $\partial\Omega$ in (6.9), it is natural to have $\mathbf{b} \in C(\overline{\Omega}; \mathbb{R}^n)$. It would be interesting to obtain the conclusion of Theorem 1.3 for such $\mathbf{b}$ instead of $\mathbf{b} \in C^1(\overline{\Omega}; \mathbb{R}^n)$.

7.2. Global Hölder estimates for pointwise Hölder continuous solutions at the boundary

In this section, we prove Theorem 3.2. The proof is similar to that of [17, Theorem 1.4] for the case without drift. For completeness, we include the proof which includes the following ingredients: interior Hölder estimates for linearized Monge–Ampère equations with bounded drifts, and rescalings using a consequence of the boundary localization theorem for the Monge–Ampère equation which we will recall below.

Under the assumption $\lambda \le \det D^2 u \le \Lambda$, the linearized Monge–Ampère operator $U^{ij} D_{ij}$ is elliptic, but it may be degenerate and singular in the sense that the eigenvalues of $U = (U^{ij})$ may tend to zero or infinity. To prove estimates for the linearized Monge–Ampère equation that are independent of the bounds on the eigenvalues of U, as in [3] and subsequent works, we work with *sections* of u instead of Euclidean balls. For a convex function $u \in C^1(\overline{\Omega})$ defined on the closure of a convex, bounded domain $\Omega \subset \mathbb{R}^n$, the section of u centered at $x \in \overline{\Omega}$ with height $h > 0$ is defined by

$$S_u(x, h) := \{y \in \overline{\Omega} : u(y) < u(x) + Du(x) \cdot (y - x) + h\}.$$

Before proving a global Hölder estimate, we recall the interior Hölder estimate. The following interior Hölder estimate for nonhomogeneous linearized Monge–Ampère equations with drift is a simple consequence of the interior Harnack inequality proved in [20, Theorem 1.1]. In [29], Maldonado proved a similar Harnack's inequality for linearized Monge–Ampère equation with drift terms with different and stronger conditions on $\mathbf{b}$.

Theorem 7.3 (Interior Hölder estimate for nonhomogeneous linearized Monge–Ampère equations with drift terms [20]). *Suppose that $u \in C^2(\Omega)$ is a strictly convex function in a bounded domain $\Omega \subset \mathbb{R}^n$ with section $S_u(0,1)$ satisfying*

$$B_{r_1}(0) \subset S_u(0,1) \subset B_{r_2}(0)$$

for some positive constants $r_1 \le r_2$, and with Hessian determinant satisfying

$$\lambda \le \det D^2 u \le \Lambda \quad \text{in } \Omega,$$

where λ and Λ are positive constants. Let $(U^{ij}) := (\det D^2u)(D^2u)^{-1}$. Let $\mathbf{b} : S_u(0,1) \to \mathbb{R}^n$ be a vector field such that $\|\mathbf{b}\|_{L^\infty(S_u(0,1))} \le M$. Let $v \in W^{2,n}_{\text{loc}}(S_u(0,1))$ be a solution to

$$U^{ij} D_{ij} v + \mathbf{b} \cdot Dv = f \quad \text{in } S_u(0,1).$$

Then there exist constants $\beta_0, C > 0$ depending only λ, Λ, n, r_1, r_2 and M such that

$$|v(x) - v(y)| \le C|x-y|^{\beta_0}(\|v\|_{L^\infty(S_u(0,1))} + \|f\|_{L^n(S_u(0,1))})$$

for all $x, y \in S_u(0,1/2)$.

To bridge the interior Hölder estimates in Theorem 7.3 and the boundary Hölder estimates in (3.3), we need to control the shape of sections of the convex function u that are tangent to the boundary $\partial\Omega$. The following proposition, proved by Savin [33] (see also [27, Proposition 3.2]), provides such a tool. It is a consequence of the boundary localization theorem for the Monge–Ampère equation, proved by Savin [32, Theorem 2.1], [34, Theorem 3.1].

Proposition 7.4 (Shape of sections tangent to the boundary, [33]). *Assume that $\Omega \subset \mathbb{R}^n$ is a uniformly convex domain with $\partial\Omega \in C^3$. Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a convex function satisfying*

$$\lambda \le \det D^2 u \le \Lambda \quad \text{in } \Omega$$

for some positive constants λ and Λ. Moreover, assume that $u|_{\partial\Omega} \in C^3$. Assume that for some $y \in \Omega$ the section $S_u(y,h) \subset \Omega$ is tangent to $\partial\Omega$ at some point $x_0 \in \partial\Omega$, that is, $\partial S_u(y,h) \cap \partial\Omega = x_0$ for some $h \le h_0(\lambda, \Lambda, \Omega, u|_{\partial\Omega}, n)$. Then there exists a small positive constant k_0 depending on λ, Λ, $\Omega, u|_{\partial\Omega}$ and n such that

$$k_0 E_h \subset S_u(y,h) - y \subset k_0^{-1} E_h, \quad k_0 h^{1/2} \le \text{dist}(y, \partial\Omega) \le k_0^{-1} h^{1/2},$$

where $E_h := h^{1/2} A_h^{-1} B_1(0)$ is an ellipsoid with A_h being a linear transformation with

$$\|A_h\|, \|A_h^{-1}\| \le k_0^{-1} |\log h|, \quad \det A_h = 1.$$

Now, we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. The equation satisfied by $(\|\varphi\|_{C^\alpha(\partial\Omega)} + \|f\|_{L^n(\Omega)})^{-1} v$ shows that we can assume that

$$\|\varphi\|_{C^\alpha(\partial\Omega)} + \|f\|_{L^n(\Omega)} = 1,$$

and we need to show that

$$\|v\|_{C^\beta(\bar{\Omega})} \leq C(\lambda, \Lambda, n, \alpha, \Omega, u|_{\partial\Omega}, \gamma, \delta, K, M)$$

for some $\beta \in (0,1)$ depending on $n, \lambda, \Lambda, \Omega, u|_{\partial\Omega}, \gamma$ and M.

Step 1: Hölder estimates in the interior of a section tangent to the boundary. Let $y \in \Omega$ with

$$r = r_y := \mathrm{dist}(y, \partial\Omega) \leq c_1(n, \lambda, \Lambda, \Omega, u|_{\partial\Omega}),$$

and consider the maximal interior section $S_u(y,h)$ centered at y, that is,

$$h = h_y := \sup\{t : S_u(y,t) \subset \Omega\}.$$

By Proposition 7.4 applied at the point $x_0 \in \partial S_u(y,h) \cap \partial\Omega$, we can find a constant $k_0(n, \lambda, \Lambda, \Omega, u|_{\partial\Omega}) > 0$ such that

$$k_0 h^{1/2} \leq r \leq k_0^{-1} h^{1/2}, \tag{7.1}$$

and $S_u(y,h)$ is equivalent to an ellipsoid E_h, that is,

$$k_0 E_h \subset S_u(y,h) - y \subset k_0^{-1} E_h,$$

where

$$E_h := h^{1/2} A_h^{-1} B_1(0) \quad \text{with} \quad \|A_h\|, \|A_h^{-1}\| \leq k_0^{-1} |\log h|, \quad \det A_h = 1. \tag{7.2}$$

Let

$$T\tilde{x} := y + h^{1/2} A_h^{-1} \tilde{x}.$$

We rescale u by setting

$$\tilde{u}(\tilde{x}) := \frac{1}{h}[u(T\tilde{x}) - u(y) - Du(y) \cdot (T\tilde{x} - y)].$$

Then

$$\lambda \leq \det D^2 \tilde{u}(\tilde{x}) \leq \Lambda,$$

and

$$B_{k_0}(0) \subset \tilde{S}_1 \subset B_{k_0^{-1}}(0), \quad \tilde{S}_1 := S_{\tilde{u}}(0,1) = h^{-1/2} A_h (S_u(y,h) - y). \tag{7.3}$$

Define the rescalings $\tilde{v}$ for v, $\tilde{\mathbf{b}}$ for $\mathbf{b}$, and $\tilde{g}$ for g by

$$\tilde{v}(\tilde{x}) := v(T\tilde{x}) - v(x_0), \quad \tilde{\mathbf{b}}(\tilde{x}) := h^{1/2} A_h \mathbf{b}(T\tilde{x}), \quad \tilde{g}(\tilde{x}) := h g(T\tilde{x}), \quad \tilde{x} \in \tilde{S}_1.$$

Simple computations give

$$D\tilde{v}(\tilde{x}) = h^{1/2} (A_h^{-1})^t Dv(T\tilde{x}),$$
$$D^2\tilde{u}(\tilde{x}) = (A_h^{-1})^t D^2 u(T\tilde{x}) A_h^{-1}, \quad D^2\tilde{v}(\tilde{x}) = h (A_h^{-1})^t D^2 v(T\tilde{x}) A_h^{-1},$$

and the cofactor matrix $\tilde{U} = (\tilde{U}^{ij})$ of $D^2\tilde{u}$ satisfies

$$\tilde{U}(\tilde{x}) := (\det D^2\tilde{u})(D^2\tilde{u})^{-1} = (\det D^2 u)A_h(D^2u)^{-1}(A_h)^t = A_h U(T\tilde{x})(A_h)^t.$$

Therefore, we find that

$$\tilde{U}^{ij}D_{ij}\tilde{v} = \operatorname{trace}(\tilde{U}D^2\tilde{v}) = h(U^{ij}D_{ij}v)(T\tilde{x}) \quad \text{in } \tilde{S}_1.$$

It is now easy to see that $\tilde{v}$ solves

$$\tilde{U}^{ij}D_{ij}\tilde{v} + \tilde{\mathbf{b}}\cdot D\tilde{v} = \tilde{g} \quad \text{in } \tilde{S}_1.$$

Due to (7.2), and the smallness of h (see (7.1)), we have the bound

$$\|\tilde{\mathbf{b}}\|_{L^\infty(\tilde{S}_1)} \leq k_0^{-1}h^{1/2}|\log h|\cdot\|\mathbf{b}\|_{L^\infty(S_u(y,h))} \leq k_0^{-1}h^{1/2}|\log h|M \leq M.$$

Now, we apply the interior Hölder estimates in Theorem 7.3 to $\tilde{v}$ to obtain a small constant $\beta \in (0,1)$ depending only on n, λ, Λ, k_0 and M such that

$$|\tilde{v}(\tilde{z}_1) - \tilde{v}(\tilde{z}_2)| \leq C_1(n,\lambda,\Lambda,M)|\tilde{z}_1 - \tilde{z}_2|^\beta\{\|\tilde{v}\|_{L^\infty(\tilde{S}_1)} + \|\tilde{g}\|_{L^n(\tilde{S}_1)}\}$$

for all $\tilde{z}_1, \tilde{z}_2 \in \tilde{S}_{1/2} := S_{\tilde{u}}(0, 1/2)$. By (7.3), we can decrease β in the above inequality if necessary, and thus assume that

$$2\beta \leq \gamma.$$

A simple computation using (7.2) gives

$$\|\tilde{g}\|_{L^n(\tilde{S}_1)} = h^{1/2}\|g\|_{L^n(S_u(y,h))}.$$

Moreover, from (7.1) and (7.2), we infer the following inclusions regarding sections and balls:

$$B_{c_2 r/|\log r|}(y) \subset S_u(y, h/2) \subset S_u(y,h) \subset B_{C_2 r|\log r|}(y) \tag{7.4}$$

for some $c_2 \in (0,1)$ and $C_2 > 0$ depending on $n, \lambda, \Lambda, \Omega, u|_{\partial\Omega}$. We also deduce that

$$\operatorname{diam}(S_u(y,h)) \leq C(n,\lambda,\Lambda,\Omega,u|_{\partial\Omega})r|\log r| \leq \delta$$

if

$$r \leq c_3(n,\lambda,\Lambda,\Omega,u|_{\partial\Omega},\delta).$$

We now consider r satisfying the above inequality. By (3.3), we have

$$\|\tilde{v}\|_{L^\infty(\tilde{S}_1)} \leq K \operatorname{diam}(S_u(y,h))^\gamma \leq C_3(r|\log r|)^\gamma,$$

where $C_3 = C_3(n,\lambda,\Lambda,\Omega,u|_{\partial\Omega},\gamma,K)$. Hence

$$|\tilde{v}(\tilde{z}_1) - \tilde{v}(\tilde{z}_2)| \leq C_4|\tilde{z}_1 - \tilde{z}_2|^\beta\{(r|\log r|)^\gamma + h^{1/2}\|g\|_{L^n(S_u(y,h))}\}$$

for all $\tilde{z}_1, \tilde{z}_2 \in \tilde{S}_{1/2}$, where $C_4 = C_4(n,\lambda,\Lambda,\Omega,u|_{\partial\Omega},\delta,\gamma,K,M)$.

Each $z \in S_u(y, h/2)$ corresponds to a unique $\tilde{z} = T^{-1}z \in \tilde{S}_{1/2}$. Rescaling back, recalling $2\beta \leq \gamma$, and using $\tilde{z}_1 - \tilde{z}_2 = h^{-1/2}A_h(z_1 - z_2)$ and the fact that

$$\begin{aligned}|\tilde{z}_1 - \tilde{z}_2| &\leq \|h^{-1/2}A_h\|\,|z_1 - z_2| \\ &\leq k_0^{-1}h^{-1/2}|\log h|\,|z_1 - z_2| \leq C_5(n, \lambda, \Lambda, \Omega, u|_{\partial\Omega})r^{-1}|\log r|\,|z_1 - z_2|,\end{aligned}$$

we find

$$|v(z_1) - v(z_2)| \leq |z_1 - z_2|^\beta \quad \text{for all } z_1, z_2 \in S_u(y, h/2), \tag{7.5}$$

provided that $r = r_y \leq c_3 < 1$ is small.

Step 2: Global Hölder estimates. We now combine (7.5) with (3.3) and (7.4) to prove

$$\|v\|_{C^\beta(\bar{\Omega})} \leq C(n, \lambda, \Lambda, \Omega, u|_{\partial\Omega}, \alpha, \delta, \gamma, K, M).$$

Indeed, as in (3.4), there exists a constant $C_*(n, \lambda, M, \mathrm{diam}(\Omega))$ such that

$$\|v\|_{L^\infty(\Omega)} \leq C_*. \tag{7.6}$$

It remains to estimate $|v(x) - v(y)|/|x - y|^\beta$ for x and y in Ω. Let $r_x = \mathrm{dist}(x, \partial\Omega)$ and $r_y = \mathrm{dist}(y, \partial\Omega)$. Assume, without loss of generality, that $r_y \leq r_x$. Take $x_0, y_0 \in \partial\Omega$ such that

$$r_x = |x - x_0| \quad \text{and } r_y = |y - y_0|.$$

From (7.6) and the interior Hölder estimates in Theorem 7.3, we only need to consider the case $r_y \leq r_x \leq c_3 < 1$. Consider the following cases.

Case 1: $|x - y| \leq c_2 r_x/|\log r_x|$. In this case, by (7.4), we have

$$y \in B_{c_2 r_x/|\log r_x|}(x) \subset S_u(x, h_x/2),$$

where

$$h_x := \sup\{t : S_u(x, t) \subset \Omega\}.$$

In view of (7.5), we have

$$\frac{|v(x) - v(y)|}{|x - y|^\beta} \leq 1.$$

Case 2: $|x - y| \geq c_2 r_x/|\log r_x|$. In this case, we have

$$r_x \leq c_2^{-1}|x - y|\,\big|\log|x - y|\big|. \tag{7.7}$$

Indeed, if

$$1 > r_x \geq |x - y|\,\big|\log|x - y|\big| \geq |x - y|$$

then

$$r_x \leq \frac{1}{c_2}|x - y|\,|\log r_x| \leq \frac{1}{c_2}|x - y|\,\big|\log|x - y|\big|.$$

Due to (7.7), we have

$$|x_0 - y_0| \leq r_x + |x - y| + r_y \leq C_6(n, \lambda, \Lambda, \Omega, u|_{\partial\Omega})|x - y|\,\big|\log|x - y|\big|.$$

Therefore, by (3.3), $\|\varphi\|_{C^\alpha(\partial\Omega)} \le 1$ and $2\beta \le \gamma \le \alpha$, we obtain

$$\begin{aligned} |v(x) - v(y)| &\le |v(x) - v(x_0)| + |v(x_0) - v(y_0)| + |v(y_0) - v(y)| \\ &\le C(r_x^\gamma + |x_0 - y_0|^\alpha + r_y^\gamma) \\ &\le C\big(|x - y|\,\big|\log|x - y|\big|\big)^\gamma \le C|x - y|^\beta, \end{aligned}$$

where $C = C(n, \lambda, \Lambda, \alpha, \Omega, u|_{\partial\Omega}, \delta, \gamma, K, M)$. This gives the desired estimate for $|v(x) - v(y)|/|x - y|^\beta$ in Case 2.

The proof of the theorem is complete. ∎

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