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Calculus of Variations

Interior Harnack inequality and Hölder estimates for linearized Monge–Ampère equations in divergence form with Drift

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Abstract

In this paper, we study interior estimates for solutions to linearized Monge–Ampère equations in divergence form with drift terms and the right-hand side containing the divergence of a bounded vector field. Equations of this type appear in the study of semigeostrophic equations in meteorology and the solvability of singular Abreu equations in the calculus of variations with a convexity constraint. We prove an interior Harnack inequality and Hölder estimates for solutions to equations of this type in two dimensions, and under an integrability assumption on the Hessian matrix of the Monge–Ampère potential in higher dimensions. Our results extend those of Le (Graduate studies in mathematics, vol 240, American Mathematical Society, 2024) to equations with drift terms.

Mathematics Subject Classification 35J15 · 35J70 · 35J75

1 Introduction and statements of the main results

In this paper, we are interested in the interior estimates for solutions $u : \Omega \to \mathbb{R}$ to linearized Monge–Ampère equations of the form

$$-\operatorname{div}(\Phi Du + u\mathbf{B}) + \mathbf{b} \cdot Du = f - \operatorname{div} \mathbf{F} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, where $\mathbf{b}$, $\mathbf{B}$, $\mathbf{F}$: $\Omega \to \mathbb{R}^n$ are bounded vector fields, $f \in L^n$, and

$$\Phi = (\Phi^{ij})_{1\leq i,j\leq n} = (\det D^2\varphi)(D^2\varphi)^{-1} \tag{1.2}$$

is the cofactor matrix of the Hessian matrix

$$D^2\varphi = (D_{ij}\varphi)_{1\leq i,j\leq n} = \left(\frac{\partial^2\varphi}{\partial x_i \partial x_j}\right)_{1\leq i,j\leq n}.$$

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Here φ is a C^3 convex Monge–Ampère potential satisfying

$$0 < \lambda \leq \det D^2\varphi \leq \Lambda \quad \text{in } \Omega. \tag{1.3}$$

As the cofactor matrix Φ is divergence-free, that is, $D_i \Phi^{ij} = 0$ for all j, the left-hand side of (1.1) can also be written in nondivergence form and we have

$$-\Phi^{ij} D_{ij} u + (\mathbf{b} - \mathbf{B}) \cdot Du - (\operatorname{div} \mathbf{B})u = f - \operatorname{div} \mathbf{F}.$$

We will focus on the divergence form and the case when $\mathbf{F} \neq 0$, and obtain interior estimates for u using its integral information.

1.1 Linearized Monge–Ampère equations

Linearized Monge–Ampère equations arise in several contexts such as affine maximal surface equation in affine geometry [50–52], Kähler metrics of constant scalar curvature in complex geometry [9, 10, 13, 14], solvability of Abreu type equations in complex geometry and in the calculus of variations with a convexity constraint [1, 7, 8, 23, 28, 29, 35, 54, 55], and semigeostrophic equations in meteorology [2, 15, 26, 36].

For a strictly convex function $\varphi \in C^2(\Omega)$ satisfying (1.3), the cofactor matrix Φ is positive definite, but we cannot expect structural bounds on its eigenvalues. Hence, the linearized Monge–Ampère operator is an elliptic operator that can be degenerate and singular.

Starting with the seminal result of Caffarelli-Gutiérrez [6] on the homogeneous equation

$$\Phi^{ij} D_{ij} u = \operatorname{div}(\Phi Du) = 0,$$

linearized Monge–Ampère equations have been studied by many authors [17, 18, 25, 30, 32–34, 37–39, 45, 47]. The term $\operatorname{div} \mathbf{F}$ in (1.1) appears in the study of semigeostrophic equations in meteorology. Specifically, we have equations of the form

$$\operatorname{div}(\Phi Du) = \operatorname{div} \mathbf{F}. \tag{1.4}$$

See [36, equation (13)], [26, equation (1.5)] and [31, equation (15.51)].

For equations of this type, Loeper [36] proved the Hölder estimate of solutions using integral information of u under the assumption that $\det D^2\varphi$ is close to a constant. Roughly speaking, Loeper needed this condition to apply the results of Murty-Stampacchia [43] and Trudinger [49]; see Sect. 1.3 for more information. Le [26] proved the same result when $n = 2$ with just the assumption in (1.3). Le [31, Theorem 15.6] also proved the Hölder estimate when $n \geq 3$ under an integrability assumption on the Hessian matrix $D^2\varphi$, that is, $D^2\varphi \in L^s$ for $s > n(n-1)/2$. This equation was also studied by Wang [53], where the Hölder estimate is proved under an integrability assumption on $(D^2\varphi)^{1/2}\mathbf{F}$; more precisely, when $(D^2\varphi)^{1/2}\mathbf{F} \in L^q$, $q > n$. In Wang [53], the upper bound for the Hölder norm contains the L^∞ norm of the solution u, while in Le [31], the L^p norm ($p > 1$) is used.

The main difference between (1.1) and (1.4) is the existence of drift terms $-\operatorname{div}(u\mathbf{B})$ and $\mathbf{b} \cdot Du$. When $\mathbf{F} = 0$, equations of the form (1.1) with nonzero drift terms ($\mathbf{b}, \mathbf{B} \neq 0$) have been studied by Maldonado [37, 40, 41] and Le [27, 30]. These appear in the solvability of singular Abreu equations in higher dimensions in complex geometry and in the calculus of variations with a convexity constraint [23, equations (2.2) and (2.5)].

1.2 The main results

In this paper, we will consider equations of type (1.1) that have both the drift terms, and also div $\mathbf{F}$, in dimension two and under an integrability assumption on $D^2\varphi$ in higher dimensions. Our main results are the following theorems on interior Harnack inequality and Hölder estimates. They extend the result of Le [31] to equations with drift terms.

Our first result is the following Harnack inequality.

Theorem 1.1 *(Harnack inequality) Let $\varphi \in C^3(\Omega)$ be a convex function satisfying (1.3). Suppose that $\mathbf{F}, \mathbf{b}, \mathbf{B} \in W^{1,n}_{\text{loc}}(\Omega; \mathbb{R}^n) \cap L^\infty_{\text{loc}}(\Omega; \mathbb{R}^n)$, $f \in L^n_{\text{loc}}(\Omega)$, $n/2 < r \leq n$, and $\operatorname{div} \mathbf{B} \leq 0$. Assume that $S_\varphi(x_0, 2h) \Subset \Omega$, where $S_\varphi(x_0, \cdot)$ is the section defined in Definition 2.1. Let $u \in W^{2,n}(S_\varphi(x_0, h))$ be a nonnegative solution to (1.1) in $S_\varphi(x_0, h)$ and let $t \leq h/2$. Further assume that*

1. *either $n = 2$, or*
2. *$n \geq 3$ and $\varepsilon^*(n, \lambda, \Lambda) + 1 > \frac{n}{2}$, where ε^* is the exponent in the interior $W^{2,1+\varepsilon}$ estimate for the Monge–Ampère equation in Theorem 2.7.*

Then, there are positive constants C and γ such that

$$\sup_{S_\varphi(x_0,t)} u \leq C \left((\|\mathbf{F}\|_{L^\infty(S_\varphi(x_0,h))} + \|f\|_{L^r(S_\varphi(x_0,h))}) t^\gamma + \inf_{S_\varphi(x_0,t)} u \right).$$

Here the constants C and γ are given by

$$\begin{aligned} &\gamma = \gamma(n, \lambda, \Lambda, r) > 0, \quad \text{and} \\ &C = C(n, \lambda, \Lambda, r, \varepsilon^*, \|\mathbf{b}\|_{L^\infty(S_\varphi(x_0,h))}, \|\mathbf{B}\|_{L^\infty(S_\varphi(x_0,h))}, \\ &\qquad \|\operatorname{div} \mathbf{B}\|_{L^n(S_\varphi(x_0,h))}, h, \operatorname{diam}(S_\varphi(x_0, 2h))). \end{aligned}$$

We will prove Theorem 1.1 in Sect. 4.

From the Harnack inequality, we have the following interior Hölder estimates.

Corollary 1.2 *(Hölder estimates with L^∞ norms) Let $\varphi \in C^3(\Omega)$ be a convex function satisfying (1.3). Assume that $\mathbf{F}, \mathbf{B}, \mathbf{b} \in L^\infty_{\text{loc}}(\Omega; \mathbb{R}^n) \cap W^{1,n}_{\text{loc}}(\Omega; \mathbb{R}^n)$, $f \in L^n_{\text{loc}}(\Omega)$, $\operatorname{div} \mathbf{B} \leq 0$, $n/2 < r \leq n$, and $S_\varphi(x_0, 4h_0) \Subset \Omega$. Let $u \in W^{2,n}_{\text{loc}}(S_\varphi(x_0, 4h_0))$ be a solution to (1.1) in $S_\varphi(x_0, 4h_0)$. Further assume that*

1. *either $n = 2$, or*
2. *$n \geq 3$ and $\varepsilon^*(n, \lambda, \Lambda) + 1 > \frac{n}{2}$, where ε^* is the exponent in the interior $W^{2,1+\varepsilon}$ estimate for the Monge–Ampère equation in Theorem 2.7.*

Then, there are positive constants C and γ such that for all $x, y \in S_\varphi(x_0, h_0)$, we have

$$|u(x) - u(y)| \leq C \left(\|\mathbf{F}\|_{L^\infty(S_\varphi(x_0,2h_0))} + \|f\|_{L^r(S_\varphi(x_0,2h_0))} + \|u\|_{L^\infty(S_\varphi(x_0,h_0))} \right) |x - y|^\gamma. \tag{1.5}$$

Here γ depends on n, λ, Λ, ε^*, $\|(\mathbf{b}, \mathbf{B})\|_{L^\infty(S_\varphi(x_0,2h_0))}$, $\|\operatorname{div} \mathbf{B}\|_{L^n(S_\varphi(x_0,2h_0))}$, $\operatorname{diam}(S_\varphi(x_0, 4h_0))$, and h_0, and C depends on $\|(\mathbf{b}, \mathbf{B})\|_{L^\infty(S_\varphi(x_0,2h_0))}$, $\|\operatorname{div} \mathbf{B}\|_{L^n(S_\varphi(x_0,2h_0))}$, $\operatorname{diam}(S_\varphi(x_0, 4h_0))$, n, λ, Λ, r, ε^*, and h_0.

We will prove Corollary 1.2 in Sect. 6.

With stronger assumptions on the integrability of the Hessian matrix $D^2\varphi$ in higher dimensions, we can obtain the following interior Hölder estimate, where the L^∞ norm of the solution u in Corollary 1.2 is replaced by its L^2 norm.

Theorem 1.3 *(Hölder estimates with L^2 norm) Let $\varphi \in C^3(\Omega)$ be a convex function satisfying (1.3). Assume that $\mathbf{F}, \mathbf{B}, \mathbf{b} \in L^\infty_{\text{loc}}(\Omega; \mathbb{R}^n) \cap W^{1,n}_{\text{loc}}(\Omega; \mathbb{R}^n)$, $f \in L^n_{\text{loc}}(\Omega)$, $\operatorname{div} \mathbf{B} \leq 0$, $n/2 < r \leq n$, and $S_\varphi(x_0, 4h_0) \Subset \Omega$. Let $u \in W^{2,n}_{\text{loc}}(S_\varphi(x_0, 4h_0))$ be a solution to (1.1) in $S_\varphi(x_0, 4h_0)$. Further assume that*

1. *either $n = 2$, or*
2. *$n \geq 3$ and $\varepsilon^*(n, \lambda, \Lambda) + 1 > \frac{n(n-1)}{2}$, where ε^* is the exponent in the interior $W^{2,1+\varepsilon}$ estimate for the Monge–Ampère equation in Theorem 2.7.*

Then, there are positive constants C and γ, where γ depends on n, λ, Λ, ε^*, $\operatorname{diam}(S_\varphi(x_0, 4h_0))$, $\|(\mathbf{b}, \mathbf{B})\|_{L^\infty(S_\varphi(x_0,2h_0))}$, and h_0, and C depends on n, λ, Λ, r, ε^*, $\|(\mathbf{b}, \mathbf{B})\|_{L^\infty(S_\varphi(x_0,2h_0))}$, h_0, and $\operatorname{diam}(S_\varphi(x_0, 4h_0))$, such that for all $x, y \in S_\varphi(x_0, h_0)$, we have

$$|u(x) - u(y)| \leq C \left(\|\mathbf{F}\|_{L^\infty(S_\varphi(x_0,2h_0))} + \|f\|_{L^r(S_\varphi(x_0,2h_0))} + \|u\|_{L^2(S_\varphi(x_0,2h_0))} \right) |x - y|^\gamma. \tag{1.6}$$

We will prove Theorem 1.3 in Sect. 6.

Remark 1.4 In Theorem 1.3, we use the L^2 norm of the solution u in the estimate (in fact, any L^p norm for $p > 0$ can be used); in Corollary 1.2, the L^∞ norm of u is used in the estimate. The improvement in Theorem 1.3 comes at the cost of having to assume stronger integrability of $D^2\varphi$ when $n \geq 3$, namely, $1 + \varepsilon^* > \frac{n(n-1)}{2}$. This is because we need this condition in the proof of the interior estimate in Lemma 5.1. It would be interesting to see if the condition $1 + \varepsilon^* > \frac{n(n-1)}{2}$ can be relaxed in Theorem 1.3.

Remark 1.5 Note that, by Caffarelli [4] (also see [31, Theorem 6.13]), for any $p > 1$ and any convex function φ satisfying (1.3), we have $D^2\varphi \in L^p_{\text{loc}}(\Omega)$, provided that $\Lambda/\lambda - 1 \leq e^{-C(n)p}$ for some large constant $C(n) > 1$.

Remark 1.6 In our theorems, we require φ to be C^3 in the domain. However, our estimates do not depend on the regularity of φ but only on the constants λ, Λ, and n. The functions $\mathbf{F}, \mathbf{b}, \mathbf{B}$ are assumed to be in $L^\infty_{\text{loc}}(\Omega; \mathbb{R}^n) \cap W^{1,n}_{\text{loc}}(\Omega; \mathbb{R}^n)$ and f to be in $L^n_{\text{loc}}(\Omega)$, but the estimates depend only on the quantities stated.

1.3 Related results for equations in divergence form

Divergence form equations

$$-\operatorname{div}(aDu + u\mathbf{B}) + \mathbf{b} \cdot Du + cu = f - \operatorname{div} \mathbf{F} \quad \text{in } \Omega \subset \mathbb{R}^n \tag{1.7}$$

have been studied in the case when the symmetric coefficient matrix $a = a(x)$ is not uniformly elliptic, but instead satisfies

$$\rho(x) I_n \leq a(x) \leq \mu(x) I_n$$

for nonnegative functions ρ and μ, where I_n is the $n \times n$ identity matrix. Murty-Stampacchia [43] and Trudinger [49] proved L^∞ and Hölder estimates for solutions to equations of the form (1.7) with integrability assumptions on μ and ρ^{-1}. Specifically, it is assumed that $\mu \in L^p$ and $\rho^{-1} \in L^q$, where $\frac{1}{p} + \frac{1}{q} < \frac{2}{n}$. These extend the classical results of De Giorgi [11], Nash [44], and Moser [42] for uniformly elliptic equations, when ρ and μ are positive constants.

Bella-Schäffner [3] extended the above results in the case of equations of the form

$$-\operatorname{div}(aDu) = 0$$

in $\Omega \subset \mathbb{R}^n$, under the assumption that $\frac{1}{p} + \frac{1}{q} < \frac{2}{n-1}$. This result is essentially sharp, as Franchi-Serapioni-Serra Cassano [16, Theorem 2] proved that a counterexample exists if $n \geq 4$ and $\frac{1}{p} + \frac{1}{q} > \frac{2}{n-1}$.

In the case when the matrix $a = \Phi$ is the cofactor matrix of the Hessian matrix $D^2\varphi$, where φ satisfies (1.3), we have

$$a = (\det D^2\varphi)(D^2\varphi)^{-1} \geq \frac{\det D^2\varphi}{\|D^2\varphi\|} I_n.$$

As $\det D^2\varphi \geq \lambda$ and $D^2\varphi \in L^{1+\varepsilon^*}$ by the $W^{2,1+\varepsilon}$ estimate for Monge–Ampère equations (see Theorem 2.7), $\rho^{-1} \in L^{1+\varepsilon^*}$. Furthermore, $\rho^{n-1}\mu$ is bounded by (1.3), and thus $\mu \in L^{(1+\varepsilon^*)/(n-1)}$. Therefore, we get

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{1+\varepsilon^*} + \frac{n-1}{1+\varepsilon^*} = \frac{n}{1+\varepsilon^*}.$$

Note that, with only the assumption that $\varepsilon^* > 0$, this is smaller than $\frac{2}{n-1}$ only when $n = 2$. When $n \geq 3$, the assumption $1+\varepsilon^* > n(n-1)/2$ in Theorem 1.3 and in Le [31] corresponds to $\frac{n}{1+\varepsilon^*} < \frac{2}{n-1}$. Compared to the results of Bella-Schäffner, these cover the equations with nonzero right-hand side (especially the case when $\mathbf{F} \neq 0$), with the assumption that the matrix $a = (\det D^2\varphi)(D^2\varphi)^{-1}$.

1.4 Methods of the proofs

We briefly discuss the differences in the proofs of the results in this paper, the results of Le [26, 31], and the results of Wang [53].

The proof of interior Hölder estimates in Le [26, 31] used the fine properties of the Green's function for the linearized Monge–Ampère operator [24, 25]. Other tools used in the proof are De Philippis-Figalli-Savin and Schmidt's $W^{2,1+\varepsilon}$ estimate [12, 46] in the case $n = 2$, and the Monge–Ampère Sobolev inequality. The $W^{2,1+\varepsilon}$ estimate is replaced by an integrability assumption for $D^2\varphi$ when $n \geq 3$. The results for the Green's function for the linearized Monge–Ampère operator with drift terms are not available, so we take an alternative approach in our proofs.

Wang [53] uses the De Giorgi iteration technique, in addition to the Monge–Ampère Sobolev inequality, in the proof of interior Hölder estimates. We will use the Moser iteration techniques similar to the ones in Gilbarg-Trudinger [20, Chapter 8] and Trudinger [49], and the Monge–Ampère Sobolev inequality in our proofs.

The rest of this paper is organized as follows. In Sect. 2, we present definitions and prior results used in the proofs of the results. In Sect. 3, we establish global L^∞ estimates for solutions to (1.1). In Sect. 4, we prove the interior Harnack inequality in Theorem 1.1. In Sect. 5, we establish interior estimates for solutions to (1.1). Finally, in Sect. 6, we prove the Hölder estimates in Corollary 1.2 and Theorem 1.3.

2 Preliminaries

In this section, we introduce some notations, definitions, and background results on the Monge–Ampère equations and the linearized Monge–Ampère equations that will be used in this paper.

Notation

We will use the following notations throughout the paper.

- $B_r(x) := \{y \in \mathbb{R}^n : |y - x| < r\}$,
- $B_r := B_r(0)$,
- $u^{\pm} := \max\{\pm u, 0\}$,
- $I_n := n \times n$ identity matrix.
- $\operatorname{diam}(E) :=$ diameter of a set E.
- $|\Omega| :=$ the Lebesgue measure of a Lebesgue measurable set $\Omega \subset \mathbb{R}^n$.

Unless otherwise stated, our convex domains are assumed to have nonempty interior.

Definition 2.1 (Sections) Let φ be a C^1 convex function in $\overline{\Omega}$. Then the *section* of φ centered at $x \in \overline{\Omega}$ with height $h > 0$ is defined as

$$S_\varphi(x, h) = \{y \in \overline{\Omega} : \varphi(y) < \varphi(x) + D\varphi(x) \cdot (y - x) + h\}.$$

Theorem 2.2 *(John's lemma [22]) Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded convex domain. Then, there is an affine transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $B_1 \subset T^{-1}\Omega \subset B_n$.*

Definition 2.3 (Normalized convex sets) An open convex set $K \subset \mathbb{R}^n$ is called *normalized* if $B_1 \subset K \subset B_n$.

We will use the following Monge–Ampère Sobolev inequality. It was proved by Tian-Wang [48, Theroem 3.1] when $n \geq 3$, and by Le [26, Proposition 2.6] when $n = 2$; see also [31, Theorem 14.15].

Theorem 2.4 *(Monge–Ampère Sobolev inequality) Let φ be a C^2 convex function satisfying (1.3), and define Φ as in (1.2). Suppose $S_\varphi(x_0, 2h) \Subset \Omega$, and $S_\varphi(x_0, h)$ is a normalized section. Then for any $u \in C_c^\infty(S_\varphi(x_0, h))$,*

$$\|u\|_{L^p(S_\varphi(x_0,h))} \leq C \left[\int_{S_\varphi(x_0,h)} \Phi Du \cdot Du \, dx \right]^{1/2},$$

where

1. $p \in (2, \infty)$ *and* $C = C(p, \lambda, \Lambda)$ *if* $n = 2$, *and*
2. $p = \frac{2n}{n-2}$ *and* $C = C(n, \lambda, \Lambda)$ *if* $n \geq 3$.

Theorem 2.5 *(Caffarelli's interior $C^{1,\alpha}$ estimate [5]) Let φ be a strictly convex solution to the Monge–Ampère equation* $\det D^2\varphi = f$ *in a convex domain $\Omega \subset \mathbb{R}^n$, where $\lambda \leq f \leq \Lambda$ for positive constants λ and Λ. If $S_\varphi(x, h) \Subset \Omega$ is a normalized section, then for all $y, z \in S_\varphi(x, h/2)$, we have*

$$|D\varphi(y) - D\varphi(z)| \leq C|y - z|^\alpha,$$

where

$$C = C(n, \lambda, \Lambda) > 0 \quad \text{and } \alpha = \alpha(n, \lambda, \Lambda) > 0. \tag{2.1}$$

This $C^{1,\alpha}$ estimate implies that sections contain balls with the same center.

Corollary 2.6 *With the same assumptions as in Theorem 2.5, if $t \le h/2$ we have*

$$B_{ct^{1/(1+\alpha)}}(x) \subset S_\varphi(x, t),$$

where α is defined in (2.1) and $c = c(n, \lambda, \Lambda) > 0$.

We will also use the interior $W^{2,1+\varepsilon}$ estimate of De Philippis-Figalli-Savin [12] and Schmidt [46] for the Monge–Ampère equation. We will use the following formulation for compactly supported sections (see [31, Corollary 6.26]).

Theorem 2.7 *(Interior $W^{2,1+\varepsilon}$ estimate) Let Ω be a convex domain in $\mathbb{R}^n$. Let φ: $\Omega \to \mathbb{R}$ be a continuous convex solution to the Monge–Ampère equation*

$$\det D^2\varphi = f \quad \textit{in } \Omega, \quad 0 < \lambda \le f \le \Lambda.$$

Suppose $S_\varphi(x_0, h)$ is a normalized section, and $S_\varphi(x_0, 2h) \Subset \Omega$. Then, for $\varepsilon = \varepsilon^*(n, \lambda, \Lambda) > 0$ and $C = C(n, \lambda, \Lambda) > 0$, we have

$$\|D^2\varphi\|_{L^{1+\varepsilon}(S_\varphi(x_0,h))} \le C.$$

We have the following volume estimates for sections (see [31, Lemma 5.6(i)]).

Lemma 2.8 *(Volume estimate for sections) Suppose φ is a C^1 convex solution to $\lambda \le \det D^2\varphi \le \Lambda$ for positive constants λ and Λ in $\Omega \subset \mathbb{R}^n$. If $S_\varphi(x, h) \Subset \Omega$, then*

$$c(\Lambda, n)h^{n/2} \le |S_\varphi(x, h)| \le C(\lambda, n)h^{n/2}$$

for positive constants c and C.

We will also use the following Harnack inequality for linearized Monge–Ampère equations with drift from Le [27, Theorem 1.1].

Theorem 2.9 *(Harnack inequality for linearized Monge–Ampère equations) Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Assume that φ satisfies (1.3), and define $\Phi = (\Phi^{ij})_{1\le i,j\le n}$ as in (1.2). Suppose that $v \ge 0$ is a $W^{2,n}_{\mathrm{loc}}(\Omega)$ solution of*

$$\Phi^{ij} D_{ij}v + \mathbf{b} \cdot Dv + cv = f \tag{2.2}$$

in a section $S := S_\varphi(x_0, 2h) \Subset \Omega$, where $h \le h_0$ for a positive, fixed h_0, $f \in L^n_{\mathrm{loc}}(\Omega)$, $c \in L^n_{\mathrm{loc}}(\Omega)$, and $\mathbf{b} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$. Then

$$\sup_{S_\varphi(x_0,h)} v \le C\left(\inf_{S_\varphi(x_0,h)} v + h^{1/2}\|f\|_{L^n(S)}\right), \tag{2.3}$$

where C is a positive constant depending on n, λ, Λ, h_0, $\|\mathbf{b}\|_{L^\infty(S)}$, and $\|c\|_{L^n(S)}$.

Definition 2.10 (Subsolutions to equation (1.1) in a domain S) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and S be a domain contained in Ω. Suppose $\mathbf{F}, \mathbf{B}, \mathbf{b} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \cap W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and $f \in L^n_{\mathrm{loc}}(\Omega)$. We say that $u \in W^{1,2}(S)$ is a *(weak) subsolution* to (1.1) if for all $v \in W^{1,2}_0(S)$ with $v \ge 0$ in S, we have

$$\int_S \Phi Du \cdot Dv\,dx + \int_S u\mathbf{B} \cdot Dv\,dx + \int_S (\mathbf{b} \cdot Du)v\,dx \le \int_S \mathbf{F} \cdot Dv\,dx + \int_S fv\,dx. \tag{2.4}$$

3 Global estimates

In this section, we prove global estimates for solutions to equation (1.1) with zero boundary data on sections in Proposition 3.5. These estimates will be used to prove the Harnack inequality, Theorem 1.1, in Sect. 4. The following is a brief outline of the steps leading to the proof of Proposition 3.5.

We begin with Lemma 3.1, which provides an estimate for subsolutions u that are nonpositive on the boundary of normalized sections. By defining suitable test functions and using Moser iteration, we derive an estimate for the L^∞ norm of u^+ in terms of its L^2 norm. In Lemma 3.2, we obtain an L^2 bound for w of the form $\log \frac{C}{C-u^+}$. Next, in Lemma 3.3 we show that w is a subsolution to a linearized Monge–Ampère equation of the form in (1.1). This gives global estimate for u^+ independent of the L^2 norm of u. Applying Lemma 3.3 to u and $-u$ gives Lemma 3.4, which provides global estimates in normalized sections. Finally, rescaling Lemma 3.4 gives us Proposition 3.5.

We now proceed with the proof of the following lemma.

Lemma 3.1 *Let $\varphi \in C^3(\Omega)$ be a convex function satisfying (1.3). Suppose $\mathbf{F}, \mathbf{B}, \mathbf{b} \in L^\infty_{\text{loc}}(\Omega; \mathbb{R}^n) \cap W^{1,n}_{\text{loc}}(\Omega; \mathbb{R}^n)$, $f \in L^n_{\text{loc}}(\Omega)$, and $n/2 < r \le n$. Suppose $S = S_\varphi(x, t)$ is a normalized section, and $S_\varphi(x, 2t) \Subset \Omega$. Suppose $u \in W^{1,2}(S) \cap C(\overline{S})$ is a subsolution to (1.1) in S satisfying $u \le 0$ on ∂S. Assume that*

1. *either $n = 2$, or*
2. *$n \ge 3$ and $\varepsilon^*(n, \lambda, \Lambda) + 1 > \frac{n}{2}$, where ε^* is as in Theorem 2.7.*

Then,

$$\sup_S u^+ \le C \left(\|\mathbf{F}\|_{L^\infty(S)} + \|f\|_{L^r(S)} + \left\|u^+\right\|_{L^2(S)} \right), \tag{3.1}$$

where

$$C = C(n, \lambda, \Lambda, r, \varepsilon^*, \|\mathbf{b}\|_{L^\infty(S)}, \|\mathbf{B}\|_{L^\infty(S)}, \left\|D^2\varphi\right\|_{L^{1+\varepsilon^*}(S)}).$$

Proof We define the test function v as in Gilbarg-Trudinger [20, Section 8.5]. Set

$$k = \|\mathbf{F}\|_{L^\infty(S)} + \|f\|_{L^r(S)},$$

and for $\beta \ge 1$ and $N \ge k$, define $H \in C^1([k, \infty))$ by

$$H(z) = \begin{cases} z^\beta - k^\beta & \text{if } k \le z \le N, \\ \beta N^{\beta-1}(z - N) + (N^\beta - k^\beta) & \text{if } N < z. \end{cases}$$

Let $w = u^+ + k \ge k$, and define

$$v = G(w) := \int_k^w |H'(s)|^2 \, ds \ge 0.$$

Then, using $v \in W^{1,2}_0(S)$ as a test function in (1.1), we get

$$\int_S \Phi Du \cdot Dv \, dx + \int_S u\mathbf{B} \cdot Dv \, dx + \int_S (\mathbf{b} \cdot Du) v \, dx \le \int_S \mathbf{F} \cdot Dv \, dx + \int_S f v \, dx. \tag{3.2}$$

Note that

1. $Dv = G'(w)Dw = H'(w)^2 Dw$,
2. v and Dv are supported on $\{u \geq 0\}$, and on the set $\{u > 0\} = \{v > 0\}$, we have $Dw = Du = Du^+$, and
3. H' is increasing on (k, ∞), hence G' is also increasing on (k, ∞). Thus,

$$G(w) = \int_k^w G'(s)\, ds \leq wG'(w).$$

Now we estimate the terms in (3.2) separately. Note that as φ is convex and $\det D^2\varphi > 0$ by (1.3), $D^2\varphi$ is positive definite. Moreover, the largest eigenvalue of $D^2\varphi$ is bounded by $\Delta\varphi$. Therefore, we have, in the sense of symmetric matrices,

$$\Phi = (\det D^2\varphi)(D^2\varphi)^{-1} \geq \frac{\det D^2\varphi}{\Delta\varphi} I_n.$$

Hence for any $\eta \in \mathbb{R}^n$, we have, by (1.3),

$$\Phi\eta \cdot \eta \geq \left(\frac{\det D^2\varphi}{\Delta\varphi}\right)|\eta|^2 \geq \left(\frac{\lambda}{\Delta\varphi}\right)|\eta|^2. \tag{3.3}$$

Using the Cauchy-Schwarz inequality and (3.3), we get

$$\begin{aligned}
-\int_S (\mathbf{b} \cdot Du)v\, dx &\leq \int_S G(w)|\mathbf{b} \cdot Dw|\, dx \leq \int_S wG'(w)|\mathbf{b} \cdot Dw|\, dx \\
&\leq \int_S \left(G'(w)\Phi Dw \cdot Dw\right)^{1/2} \left(w^2 G'(w)\frac{\Delta\varphi}{\lambda}|\mathbf{b}|^2\right)^{1/2} dx \\
&\leq \frac{1}{4}\int_S G'(w)\Phi Dw \cdot Dw\, dx + \int_S w^2 G'(w)\frac{\Delta\varphi}{\lambda}|\mathbf{b}|^2\, dx.
\end{aligned} \tag{3.4}$$

Similarly, recalling that Dv is supported on $\{u \geq 0\}$, we have

$$\begin{aligned}
-\int_S u\mathbf{B} \cdot Dv\, dx &= -\int_S G'(w)u\mathbf{B} \cdot Dw\, dx \leq \int_S G'(w)w|\mathbf{B}||Dw|\, dx \\
&\leq \int_S \left(G'(w)\Phi Dw \cdot Dw\right)^{1/2} \left(w^2 G'(w)\frac{\Delta\varphi}{\lambda}|\mathbf{B}|^2\right)^{1/2} dx \\
&\leq \frac{1}{4}\int_S G'(w)\Phi Dw \cdot Dw\, dx + \int_S w^2 G'(w)\frac{\Delta\varphi}{\lambda}|\mathbf{B}|^2\, dx.
\end{aligned} \tag{3.5}$$

By the same reason, we have

$$\begin{aligned}
\int_S \mathbf{F} \cdot Dv\, dx &= \int_S G'(w)\mathbf{F} \cdot Dw\, dx \\
&\leq \int_S \left(G'(w)\Phi Dw \cdot Dw\right)^{1/2} \left(G'(w)\frac{\Delta\varphi}{\lambda}|\mathbf{F}|^2\right)^{1/2} dx \\
&\leq \frac{1}{4}\int_S G'(w)\Phi Dw \cdot Dw\, dx + \int_S G'(w)\frac{\Delta\varphi}{\lambda}|\mathbf{F}|^2\, dx \\
&\leq \frac{1}{4}\int_S G'(w)\Phi Dw \cdot Dw\, dx + \int_S w^2 G'(w)\frac{\Delta\varphi}{\lambda}\, dx,
\end{aligned} \tag{3.6}$$

where we used $w \geq \|\mathbf{F}\|_{L^\infty(S)}$, and because $w \geq k$,

$$\begin{aligned}\int_S fv\,dx &\leq \int_S |f|G(w)\,dx \leq \int_S |f|wG'(w)\,dx \\ &\leq \int_S \frac{|f|}{k}w^2G'(w)\,dx.\end{aligned} \tag{3.7}$$

Note that

$$\int_S \Phi Du\cdot Dv\,dx = \int_S G'(w)\Phi Dw\cdot Dw.$$

Adding (3.4)–(3.7) and invoking (3.2), we obtain

$$\begin{aligned}&\int_S G'(w)\Phi Dw\cdot Dw\,dx \\ &\leq \frac{3}{4}\int_S G'(w)\Phi Dw\cdot Dw\,dx + \int_S w^2G'(w)\left[\frac{\Delta\varphi}{\lambda}(1+|\mathbf{b}|^2+|\mathbf{B}|^2)+\frac{|f|}{k}\right]dx.\end{aligned}$$

Hence

$$\int_S G'(w)\Phi Dw\cdot Dw\,dx \leq 4\int_S w^2G'(w)h\,dx, \tag{3.8}$$

where

$$h = \frac{\Delta\varphi}{\lambda}(1+|\mathbf{b}|^2+|\mathbf{B}|^2)+\frac{|f|}{k}.$$

Before moving to the next step, we estimate h. As S is normalized, $|B_1| \leq |S| \leq |B_n|$. Therefore, for

$$q := \min\{1+\varepsilon^*, r\} > \frac{n}{2}$$

we have, by the Hölder inequality,

$$\begin{aligned}\|h\|_{L^q(S)} &\leq \frac{1+\|\mathbf{b}\|^2_{L^\infty(S)}+\|\mathbf{B}\|^2_{L^\infty(S)}}{\lambda}\|\Delta\varphi\|_{L^q(S)} + \frac{\|f\|_{L^q(S)}}{k} \\ &\leq \frac{1+\|\mathbf{b}\|^2_{L^\infty(S)}+\|\mathbf{B}\|^2_{L^\infty(S)}}{\lambda}\|\Delta\varphi\|_{L^{1+\varepsilon^*}(S)}|S|^{\frac{1+\varepsilon^*-q}{q(1+\varepsilon^*)}} + \frac{\|f\|_{L^r(S)}}{k}|S|^{\frac{r-q}{qr}} \\ &\leq \frac{1+\|\mathbf{b}\|^2_{L^\infty(S)}+\|\mathbf{B}\|^2_{L^\infty(S)}}{\lambda}\|\Delta\varphi\|_{L^{1+\varepsilon^*}(S)}|B_n|^{\frac{1+\varepsilon^*-q}{q(1+\varepsilon^*)}} + |B_n|^{\frac{r-q}{qr}}.\end{aligned} \tag{3.9}$$

Then, for

$$\widehat{q} := \frac{2q}{q-1}, \tag{3.10}$$

we have, from the Hölder inequality

$$\begin{aligned}\int_S w^2G'(w)h\,dx &= \int_S (wH'(w))^2h\,dx \\ &\leq \|h\|_{L^q(S)}\left\|(wH'(w))^2\right\|_{L^{\frac{q}{q-1}}(S)} \\ &= \|h\|_{L^q(S)}\left\|wH'(w)\right\|^2_{L^{\widehat{q}}(S)}.\end{aligned} \tag{3.11}$$

As $u \leq 0$ on ∂S, $H(w) = 0$ on ∂S and the Monge–Ampère Sobolev inequality, Theorem 2.4, implies

$$\begin{aligned}\int_S G'(w)\Phi Dw \cdot Dw\, dx &= \int_S H'(w)^2 \Phi Dw \cdot Dw\, dx \\ &= \int_S \Phi DH(w) \cdot DH(w)\, dx \\ &\geq c_1(q, n, \lambda, \Lambda)\, \|H(w)\|^2_{L^{\widehat{n}}(S)},\end{aligned} \tag{3.12}$$

where

$$\widehat{n} = \begin{cases} \frac{2n}{n-2} & \text{if } n \geq 3, \\ 2\widehat{q} & \text{if } n = 2. \end{cases} \tag{3.13}$$

Note that as $q > n/2$, we have $\widehat{n} > \widehat{q}$. From (3.8), (3.11), and (3.12), we have

$$\|H(w)\|_{L^{\widehat{n}}(S)} \leq C_2(q, n, \lambda, \Lambda)\, \|h\|^{1/2}_{L^q(S)} \left\|wH'(w)\right\|_{L^{\widehat{q}}(S)}. \tag{3.14}$$

Letting $N \to \infty$, the terms in (3.14) converge to

$$\begin{aligned}\|H(w)\|_{L^{\widehat{n}}(S)} &\to \left\|w^\beta - k^\beta\right\|_{L^{\widehat{n}}(S)}, \\ \left\|wH'(w)\right\|_{L^{\widehat{q}}(S)} &\to \left\|\beta w^\beta\right\|_{L^{\widehat{q}}(S)}.\end{aligned} \tag{3.15}$$

We also have

$$\begin{aligned}\left\|k^\beta\right\|_{L^{\widehat{n}}(S)} = k^\beta |S|^{1/\widehat{n}} &= |S|^{1/\widehat{n} - 1/\widehat{q}} \left\|k^\beta\right\|_{L^{\widehat{q}}(S)} \\ &\leq |B_1|^{1/\widehat{n}-1/\widehat{q}} \left\|k^\beta\right\|_{L^{\widehat{q}}(S)} \\ &\leq |B_1|^{1/\widehat{n}-1/\widehat{q}} \left\|w^\beta\right\|_{L^{\widehat{q}}(S)}.\end{aligned} \tag{3.16}$$

Because $\beta \geq 1$, from (3.14)–(3.16) and (3.9), we conclude that

$$\left\|w^\beta\right\|_{L^{\widehat{n}}(S)} \leq \widetilde{C}\beta \left\|w^\beta\right\|_{L^{\widehat{q}}(S)}, \tag{3.17}$$

where

$$\begin{aligned}\widetilde{C} &= C_2(n, \lambda, \Lambda, q)\, \|h\|^{1/2}_{L^q(S)} + |B_1|^{1/\widehat{n}-1/\widehat{q}} \\ &\leq \widetilde{C}(n, \lambda, \Lambda, r, \varepsilon^*, \|\mathbf{b}\|_{L^\infty(S)}, \|\mathbf{B}\|_{L^\infty(S)}, \left\|D^2\varphi\right\|_{L^{1+\varepsilon^*}(S)}).\end{aligned}$$

Note that $\widetilde{C}$ is independent of β.

We define

$$\chi := \frac{\widehat{n}}{\widehat{q}} > 1,$$

and rewrite (3.17) as

$$\|w\|_{L^{\beta\chi\widehat{q}}(S)} \leq (\widetilde{C}\beta)^{1/\beta}\, \|w\|_{L^{\beta\widehat{q}}(S)}. \tag{3.18}$$

Setting $\beta = \chi^m \geq 1$ (for integer $m \geq 0$) in (3.18), we get

$$\|w\|_{L^{\chi^{m+1}\widehat{q}}(S)} \leq \widetilde{C}^{\chi^{-m}} \chi^{m\chi^{-m}}\, \|w\|_{L^{\chi^m\widehat{q}}(S)}. \tag{3.19}$$

Iterating (3.19) yields

$$\|w\|_{L^\infty(S)} \leq \widetilde{C}^{\sum_{m\geq 0}\chi^{-m}} \chi^{\sum_{m\geq 0} m\chi^{-m}}\, \|w\|_{L^{\widehat{q}}(S)}. \tag{3.20}$$

Because $w \geq u^+ \geq 0$ and

$$\|w\|_{L^{\widehat{q}}(S)} \leq \|w\|_{L^\infty(S)}^{1-2/\widehat{q}} \|w\|_{L^2(S)}^{2/\widehat{q}},$$

(3.20) gives

$$\begin{aligned}\sup_S u^+ &\leq \|w\|_{L^\infty(S)} \leq C\|w\|_{L^2(S)} \leq C(k + \|u^+\|_{L^2(S)}) \\ &= C\left(\|\mathbf{F}\|_{L^\infty(S)} + \|f\|_{L^r(S)} + \|u^+\|_{L^2(S)}\right),\end{aligned}$$

where

$$C = C(n, \lambda, \Lambda, r, \varepsilon^*, \|\mathbf{b}\|_{L^\infty(S)}, \|\mathbf{B}\|_{L^\infty(S)}, \|D^2\varphi\|_{L^{1+\varepsilon^*}(S)}).$$

This completes the proof. □

Note that the L^2 norm of u^+ appears on the right-hand side of (3.1). We will use a trick in Gilbarg-Trudinger [20, Section 8.5] to eliminate this term. We first prove the following lemma.

Lemma 3.2 *Let $\varphi \in C^3(\Omega)$ be a convex function satisfying (1.3). Suppose $\mathbf{F}, \mathbf{B}, \mathbf{b} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \cap W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$, $f \in L^n_{\mathrm{loc}}(\Omega)$, $n/2 < r \leq n$, and $\operatorname{div} \mathbf{B} \leq 0$. Suppose $S = S_\varphi(x, t)$ is a normalized section and $S_\varphi(x, 2t) \Subset \Omega$. Suppose $u \in W^{1,2}(S) \cap C(\overline{S})$ is a subsolution to (1.1) in S satisfying $u \leq 0$ on ∂S. Assume that*

1. *either $n = 2$, or*
2. *$n \geq 3$ and $\varepsilon^*(n, \lambda, \Lambda) + 1 > \frac{n}{2}$, where ε^* is as in Theorem 2.7.*

Then,

$$w = \log \frac{M + k}{M + k - u^+} \quad \text{where } M = \sup_S u^+ \quad \text{and } k = \|f\|_{L^r(S)} + \|\mathbf{F}\|_{L^\infty(S)}, \tag{3.21}$$

satisfies

$$\|w\|_{L^2(S)} \leq C(n, \lambda, \Lambda, r, \varepsilon^*, \|\mathbf{b}\|_{L^\infty(S)}, \|\mathbf{B}\|_{L^\infty(S)}, \|D^2\varphi\|_{L^{1+\varepsilon^*}(S)}). \tag{3.22}$$

Proof Set

$$v := \frac{u^+}{M + k - u^+}.$$

Then $v \geq 0$, and $v \in W^{1,2}_0(S) \cap C(\overline{S})$. Because u is a subsolution to (1.1), we get

$$\int_S \Phi Du \cdot Dv\, dx + \int_S u\mathbf{B} \cdot Dv\, dx + \int_S (\mathbf{b} \cdot Du) v\, dx \leq \int_S \mathbf{F} \cdot Dv\, dx + \int_S f v\, dx.$$

As $\operatorname{div} \mathbf{B} \leq 0$ and $uv \geq 0$,

$$\int_S u\mathbf{B} \cdot Dv\, dx = \int_S \mathbf{B} \cdot D(uv)\, dx - \int_S v\mathbf{B} \cdot Du\, dx \geq -\int_S v\mathbf{B} \cdot Du\, dx$$

and therefore, we have

$$\int_S \Phi Du \cdot Dv\, dx + \int_S ((\mathbf{b} - \mathbf{B}) \cdot Du) v\, dx \leq \int_S \mathbf{F} \cdot Dv\, dx + \int_S f v\, dx. \tag{3.23}$$

Because

$$Dv = \frac{M + k}{(M + k - u^+)^2} Du^+, \tag{3.24}$$

the left-hand side of (3.23) becomes

$$\begin{aligned}&\int_S \Phi Du\cdot Dv\,dx+\int_S((\mathbf{b}-\mathbf{B})\cdot Du)v\,dx\\&=\int_S\frac{M+k}{(M+k-u^+)^2}\Phi Du^+\cdot Du^+\,dx+\int_S\frac{u^+(\mathbf{b}-\mathbf{B})\cdot Du^+}{M+k-u^+}\,dx.\end{aligned}\tag{3.25}$$

We may also use (3.24) to substitute Dv in the right-hand side of (3.23) to obtain

$$\int_S \mathbf{F}\cdot Dv\,dx+\int_S fv\,dx=\int_S\frac{(M+k)\mathbf{F}\cdot Du^+}{(M+k-u^+)^2}\,dx+\int_S\frac{fu^+}{M+k-u^+}\,dx.\tag{3.26}$$

Putting (3.23), (3.25), and (3.26) together, and dividing both sides by $M+k$, we find

$$\int_S\frac{\Phi Du^+\cdot Du^+}{(M+k-u^+)^2}\,dx\le\int_S\frac{\mathbf{F}\cdot Du^+}{(M+k-u^+)^2}\,dx+\int_S\frac{fu^++u^+(\mathbf{B}-\mathbf{b})\cdot Du^+}{(M+k)(M+k-u^+)}\,dx.\tag{3.27}$$

Now we estimate the terms in (3.27) separately. First, from the Cauchy-Schwarz inequality and (3.3), we have

$$\begin{aligned}\int_S\frac{\mathbf{F}\cdot Du^+}{(M+k-u^+)^2}\,dx&\le\int_S\frac{(\Phi Du^+\cdot Du^+)^{1/2}(\lambda^{-1}\Delta\varphi|\mathbf{F}|^2)^{1/2}}{(M+k-u^+)^2}\,dx\\&\le\frac{1}{4}\int_S\frac{\Phi Du^+\cdot Du^+}{(M+k-u^+)^2}\,dx+\int_S\frac{\lambda^{-1}\Delta\varphi|\mathbf{F}|^2}{(M+k-u^+)^2}\,dx\\&\le\frac{1}{4}\int_S\frac{\Phi Du^+\cdot Du^+}{(M+k-u^+)^2}\,dx+\int_S\lambda^{-1}\Delta\varphi\,dx\end{aligned}\tag{3.28}$$

as $M+k-u^+\ge k\ge\|\mathbf{F}\|_{L^\infty(S)}$. Next, using $k\ge\|f\|_{L^r(S)}$ and the Hölder inequality, we estimate

$$\begin{aligned}\int_S\frac{fu^+}{(M+k-u^+)(M+k)}\,dx&\le\int_S\frac{|f|}{k}\times 1\,dx\le\left\|\frac{f}{k}\right\|_{L^r(S)}\|1\|_{L^{r/(r-1)}(S)}\\&\le|S|^{\frac{r-1}{r}}\le|B_n|^{\frac{r-1}{r}}.\end{aligned}\tag{3.29}$$

Finally, from Cauchy-Schwarz inequality and (3.3), we estimate

$$\begin{aligned}\int_S\frac{u^+(\mathbf{B}-\mathbf{b})\cdot Du^+}{(M+k)(M+k-u^+)}\,dx&\le\int_S|\mathbf{b}-\mathbf{B}|\left|\frac{Du^+}{M+k-u^+}\right|\,dx\\&\le\int_S\left\{\frac{\Phi Du^+\cdot Du^+}{(M+k-u^+)^2}\right\}^{1/2}\left\{\frac{\Delta\varphi}{\lambda}|\mathbf{b}-\mathbf{B}|^2\right\}^{1/2}\,dx\\&\le\frac{1}{4}\int_S\frac{\Phi Du^+\cdot Du^+}{(M+k-u^+)^2}\,dx+\int_S\frac{\Delta\varphi}{\lambda}|\mathbf{b}-\mathbf{B}|^2\,dx.\end{aligned}\tag{3.30}$$

Combining (3.27) with (3.28)–(3.30) yields

$$\begin{aligned}\frac{1}{2}\int_S\frac{\Phi Du^+\cdot Du^+}{(M+k-u^+)^2}\,dx&\le|B_n|^{\frac{r-1}{r}}+\frac{1+\left(\|\mathbf{b}\|_{L^\infty(S)}+\|\mathbf{B}\|_{L^\infty(S)}\right)^2}{\lambda}\int_S\Delta\varphi\,dx\\&\le C_0(n,r,\varepsilon^*,\lambda,\Lambda,\|\mathbf{b}\|_{L^\infty(S)},\|\mathbf{B}\|_{L^\infty(S)},\left\|D^2\varphi\right\|_{L^{1+\varepsilon^*}(S)}).\end{aligned}\tag{3.31}$$

As $u \leq 0$ on ∂S, $w = 0$ on ∂S. Also, we have

$$Dw = \frac{Du^+}{M + k - u^+}. \tag{3.32}$$

Therefore, the left-hand side of (3.31) can be estimated using the Hölder inequality and the Monge–Ampère Sobolev inequality in Theorem 2.4:

$$\begin{aligned} \frac{1}{2}\int_S \frac{\Phi Du^+ \cdot Du^+}{(M+k-u^+)^2}\,dx &= \frac{1}{2}\int_S \Phi Dw \cdot Dw\,dx \\ &\geq \begin{cases} c_1 \|w\|^2_{L^{\frac{2n}{n-2}}(S)} \geq c_1 |S|^{-2/n} \|w\|^2_{L^2(S)} \geq c_1 |B_n|^{-2/n} \|w\|^2_{L^2(S)} & \text{if } n \geq 3, \\ c_1 \|w\|^2_{L^4(S)} \geq c_1 |B_n|^{-1/2} \|w\|^2_{L^2(S)} & \text{if } n = 2, \end{cases} \end{aligned} \tag{3.33}$$

where $c_1 = c_1(n, \lambda, \Lambda)$. The conclusion of the lemma follows from (3.31) and (3.33). □

Now we obtain the following global L^∞ estimate, independent of the L^2 norm of the solution u, by showing that w in (3.21) is a subsolution to an equation of the same form as (1.1).

Lemma 3.3 *Let $\varphi \in C^3(\Omega)$ be a convex function satisfying (1.3). Suppose $\mathbf{F}, \mathbf{B}, \mathbf{b} \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \cap W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$, $f \in L^n_{\mathrm{loc}}(\Omega)$, $n/2 < r \leq n$, and $\operatorname{div}\mathbf{B} \leq 0$. Suppose $S = S_\varphi(x, t)$ is a normalized section and $S_\varphi(x, 2t) \Subset \Omega$. Suppose $u \in W^{1,2}(S) \cap C(\overline{S})$ is a subsolution to (1.1) in S satisfying $u \leq 0$ on ∂S. Assume that*

1. *either $n = 2$, or*
2. *$n \geq 3$ and $\varepsilon^*(n, \lambda, \Lambda) + 1 > \frac{n}{2}$, where ε^* is from Theorem 2.7.*

Then,

$$\sup_S u^+ \leq C\left(\|\mathbf{F}\|_{L^\infty(S)} + \|f\|_{L^r(S)}\right), \tag{3.34}$$

where

$$C = C(n, \lambda, \Lambda, r, \varepsilon^*, \|\mathbf{b}\|_{L^\infty(S)}, \|\mathbf{B}\|_{L^\infty(S)}, \left\|D^2\varphi\right\|_{L^{1+\varepsilon^*}(S)}).$$

Proof Let $w \in W^{1,2}_0(S)$ be as in (3.21). Then, using (3.32), we get

$$\begin{aligned} -\mathbf{B} \cdot Dw &= -\frac{\mathbf{B} \cdot Du^+}{M + k - u^+} \\ &= \frac{-\operatorname{div}(\mathbf{B}u^+) + u^+ \operatorname{div}\mathbf{B}}{M + k - u^+} \leq \frac{-\operatorname{div}(\mathbf{B}u^+)}{M + k - u^+} \end{aligned} \tag{3.35}$$

as $\operatorname{div}\mathbf{B} \leq 0$. We have in the weak sense,

$$-\operatorname{div}(\Phi Dw) = -\frac{\operatorname{div}(\Phi Du^+)}{M + k - u^+} - \frac{\Phi Du^+ \cdot Du^+}{(M + k - u^+)^2}. \tag{3.36}$$

From (3.32), (3.35), and (3.36), we get

$$-\operatorname{div}(\Phi Dw) + (\mathbf{b} - \mathbf{B}) \cdot Dw \leq \frac{-\operatorname{div}(\Phi Du^+ + u^+\mathbf{B}) + \mathbf{b} \cdot Du^+}{M + k - u^+} - \frac{\Phi Du^+ \cdot Du^+}{(M + k - u^+)^2}.$$

Combining this with (1.1), we get

$$
\begin{aligned}
&-\operatorname{div}(\Phi Dw) + (\mathbf{b} - \mathbf{B}) \cdot Dw \\
&\leq \frac{f - \operatorname{div}\mathbf{F}}{M + k - u^+} - \frac{\Phi Du^+ \cdot Du^+}{(M + k - u^+)^2} \\
&= -\operatorname{div}\left(\frac{\mathbf{F}}{M + k - u^+}\right) + \frac{f}{M + k - u^+} + \left(\frac{-\Phi Du^+ \cdot Du^+ + \mathbf{F} \cdot Du^+}{(M + k - u^+)^2}\right) \quad \text{in } \{u \geq 0\}.
\end{aligned}
$$

From (3.3) and the Cauchy-Schwarz inequality, we have in S

$$
\begin{aligned}
-\Phi Du^+ \cdot Du^+ + \mathbf{F} \cdot Du^+ &\leq -\frac{\lambda}{\Delta\varphi}|Du^+|^2 + \mathbf{F} \cdot Du^+ \\
&\leq \frac{\Delta\varphi |\mathbf{F}|^2}{4\lambda} \\
&\leq \frac{\Delta\varphi (M + k - u^+)^2}{4\lambda},
\end{aligned}
$$

which implies

$$
-\operatorname{div}(\Phi Dw) + (\mathbf{b} - \mathbf{B}) \cdot Dw \leq -\operatorname{div}\left(\frac{\mathbf{F}}{M + k - u^+}\right) + \frac{f}{M + k - u^+} + \frac{\Delta\varphi}{4\lambda} \quad \text{in } \{u \geq 0\}.
$$

As $w = 0$ outside $\{u \geq 0\}$, w is a subsolution to

$$
-\operatorname{div}(\Phi Dw) + \widetilde{\mathbf{b}} \cdot Dw \leq -\operatorname{div}\widetilde{\mathbf{F}} + \widetilde{f} \quad \text{in } S, \tag{3.37}
$$

where

$$
\begin{aligned}
\widetilde{\mathbf{b}} &= \mathbf{b} - \mathbf{B}, \\
\widetilde{\mathbf{F}} &= \frac{\mathbf{F}}{M + k - u^+}\chi_{\{u \geq 0\}}, \quad \text{and} \\
\widetilde{f} &= \left(\frac{f}{M + k - u^+} + \frac{\Delta\varphi}{4\lambda}\right)\chi_{\{u \geq 0\}}.
\end{aligned} \tag{3.38}
$$

Recalling that $k = \|\mathbf{F}\|_{L^\infty(S)} + \|f\|_{L^r(S)}$ and $M = \sup_S u^+ \geq u^+$, we obtain

$$
\|\widetilde{\mathbf{b}}\|_{L^\infty(S)} \leq \|\mathbf{b}\|_{L^\infty(S)} + \|\mathbf{B}\|_{L^\infty(S)}, \quad \text{and} \quad \|\widetilde{\mathbf{F}}\|_{L^\infty(S)} \leq 1. \tag{3.39}
$$

For $\widetilde{r} := \min\{r, 1 + \varepsilon^*\} > n/2$, using the Hölder inequality and the volume estimate in Lemma 2.8, we have

$$
\begin{aligned}
\|\widetilde{f}\|_{L^{\widetilde{r}}(S)} &\leq \frac{\|f\|_{L^{\widetilde{r}}(S)}}{k} + \frac{\|\Delta\varphi\|_{L^{\widetilde{r}}(S)}}{4\lambda} \\
&\leq C_1(n, r, \varepsilon^*)\left(\frac{\|f\|_{L^r(S)}}{k} + \frac{\|D^2\varphi\|_{L^{1+\varepsilon^*}(S)}}{4\lambda}\right) \\
&\leq C_1(n, r, \varepsilon^*)\left(1 + \frac{\|D^2\varphi\|_{L^{1+\varepsilon^*}(S)}}{4\lambda}\right) \\
&\leq C_2\left(n, r, \varepsilon^*, \lambda, \|D^2\varphi\|_{L^{1+\varepsilon^*}(S)}\right).
\end{aligned} \tag{3.40}
$$

Combining (3.37)–(3.40) and applying Lemmas 3.1 and 3.2, we get

$$\begin{aligned}\sup_S w &\le C_3\left(\|\widetilde{\mathbf{F}}\|_{L^\infty(S)} + \|f\|_{L^{\tilde r}(S)} + \|w\|_{L^2(S)}\right)\\ &\le C_4(n,\lambda,\Lambda,r,\varepsilon^*,\|\mathbf{b}\|_{L^\infty(S)},\|\mathbf{B}\|_{L^\infty(S)},\left\|D^2\varphi\right\|_{L^{1+\varepsilon^*}(S)}).\end{aligned} \tag{3.41}$$

Recalling that

$$w = \log\frac{M+k}{M+k-u^+}$$

and $M = \sup_S u^+$, we have

$$\sup_S w = \log\frac{M+k}{k}. \tag{3.42}$$

Therefore, as $k = \|\mathbf{F}\|_{L^\infty(S)} + \|f\|_{L^r(S)}$, the conclusion of the lemma follows from (3.41) and (3.42). □

By applying Lemma 3.3 to u and $-u$, we obtain the following estimate.

Lemma 3.4 *Let $\varphi \in C^3(\Omega)$ be a convex function satisfying (1.3). Suppose $\mathbf{F},\mathbf{B},\mathbf{b} \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^n)\cap W^{1,n}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$, $f \in L^n_{\mathrm{loc}}(\Omega)$, $n/2 < r \le n$, and $\operatorname{div}\mathbf{B} \le 0$. Suppose $S = S_\varphi(x,t)$ is a normalized section and $S_\varphi(x,2t) \Subset \Omega$. Suppose $u \in W^{1,2}(S)\cap C(\overline{S})$ is a solution to (1.1) in S satisfying $u = 0$ on ∂S. Assume that*

1. *either $n = 2$, or*
2. *$n \ge 3$ and $\varepsilon^*(n,\lambda,\Lambda) + 1 > \frac{n}{2}$, where ε^* is from Theorem 2.7.*

Then,

$$\|u\|_{L^\infty(S)} \le C\left(\|\mathbf{F}\|_{L^\infty(S)} + \|f\|_{L^r(S)}\right) \tag{3.43}$$

where

$$C = C(n,\lambda,\Lambda,r,\varepsilon^*,\|\mathbf{b}\|_{L^\infty(S)},\|\mathbf{B}\|_{L^\infty(S)},\left\|D^2\varphi\right\|_{L^{1+\varepsilon^*}(S)}).$$

Now, we rescale (1.1) and apply Lemma 3.4 to obtain the following global estimate.

Proposition 3.5 *(Global L^∞ estimate in normalized section) Let $\varphi \in C^3(\Omega)$ be a convex function satisfying (1.3). Suppose $\mathbf{F},\mathbf{B},\mathbf{b} \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^n)\cap W^{1,n}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$, $f \in L^n_{\mathrm{loc}}(\Omega)$, $n/2 < r \le n$, and $\operatorname{div}\mathbf{B} \le 0$. Suppose $S_\varphi(x_0,2h_0)$ is a normalized section contained in Ω, and $h \le h_0$. Assume that $u \in W^{1,2}(S)\cap C(\overline{S})$ is a solution to (1.1) in $S = S_\varphi(x_0,h)$ satisfying $u = 0$ on ∂S. Further assume that*

1. *either $n = 2$, or*
2. *$n \ge 3$ and $\varepsilon^*(n,\lambda,\Lambda) + 1 > \frac{n}{2}$, where ε^* is from Theorem 2.7.*

Then,

$$\|u\|_{L^\infty(S)} \le C\left(\|\mathbf{F}\|_{L^\infty(S)} + \|f\|_{L^r(S)}\right)h^\gamma, \tag{3.44}$$

where

$$\begin{aligned}&C = C(n,\lambda,\Lambda,r,\varepsilon^*,\|\mathbf{b}\|_{L^\infty(S)},\|\mathbf{B}\|_{L^\infty(S)}), \quad \textit{and}\\ &\gamma = \gamma(n,\lambda,\Lambda,r) > 0.\end{aligned}$$

Proof We use the rescaling in Le ([30, pp.20-22], [26, Section 3.2]). By John's lemma, there is an affine transformation $Tx = A_h x + b_h$ such that $B_1 \subset T^{-1}(S_\varphi(x_0, h)) \subset B_n$. We define the rescaled functions

$$\begin{aligned}
\widetilde{\varphi}(x) &:= (\det A_h)^{-2/n}\varphi(Tx),\\
\widetilde{u}(x) &:= u(Tx),\\
\widetilde{\mathbf{F}}(x) &:= (\det A_h)^{2/n} A_h^{-1}\mathbf{F}(Tx),\\
\widetilde{\mathbf{b}}(x) &:= (\det A_h)^{2/n} A_h^{-1}\mathbf{b}(Tx),\\
\widetilde{\mathbf{B}}(x) &:= (\det A_h)^{2/n} A_h^{-1}\mathbf{B}(Tx), \quad \text{and}\\
\widetilde{f}(x) &:= (\det A_h)^{2/n} f(Tx)
\end{aligned} \tag{3.45}$$

on

$$\widetilde{S} := T^{-1}(S_\varphi(x_0, h)) = S_{\widetilde{\varphi}}(y_0, (\det A_h)^{-2/n} h), \tag{3.46}$$

where $y_0 = T^{-1}x_0$. Then, the rescaled functions satisfy the equation

$$-\operatorname{div}(\widetilde{\Phi} D\widetilde{u} + \widetilde{u}\widetilde{\mathbf{B}}) + \widetilde{\mathbf{b}} \cdot D\widetilde{u} = \widetilde{f} - \operatorname{div}\widetilde{\mathbf{F}} \quad \text{in } \widetilde{S}. \tag{3.47}$$

To apply Lemma 3.4 to $\widetilde{u}$, we estimate the rescaled functions. First, note that

$$\det D^2\widetilde{\varphi}(x) = (\det D^2\varphi)(Tx) \quad \text{in } \widetilde{S},$$

so that

$$\lambda \le \det D^2\widetilde{\varphi} \le \Lambda \quad \text{in } \widetilde{S}.$$

Furthermore, as $B_1 \subset \widetilde{S} \subset B_n$, we have from Lemma 2.8,

$$c(n, \lambda, \Lambda)h^{n/2} \le \det A_h \le C(n, \lambda, \Lambda)h^{n/2}. \tag{3.48}$$

From Corollary 2.6, we get

$$\left\| A_h^{-1} \right\| \le \frac{n}{ch^{\frac{1}{1+\alpha}}} \le C(n, \lambda, \Lambda)h^{-\frac{1}{1+\alpha}}. \tag{3.49}$$

Now, from (3.45), (3.48), and (3.49), we get

$$\left\|\widetilde{\mathbf{b}}\right\|_{L^\infty(\widetilde{S})} \le (Ch^{n/2})^{2/n} Ch^{-\frac{1}{1+\alpha}} \|\mathbf{b}\|_{L^\infty(S)} \le C(n, \lambda, \Lambda)h^{\frac{\alpha}{1+\alpha}} \|\mathbf{b}\|_{L^\infty(S)}. \tag{3.50}$$

Similarly, we also obtain

$$\begin{aligned}
\left\|\widetilde{\mathbf{B}}\right\|_{L^\infty(\widetilde{S})} &\le C(n, \lambda, \Lambda)h^{\frac{\alpha}{1+\alpha}} \|\mathbf{B}\|_{L^\infty(S)}, \quad \text{and}\\
\left\|\widetilde{\mathbf{F}}\right\|_{L^\infty(\widetilde{S})} &\le C(n, \lambda, \Lambda)h^{\frac{\alpha}{1+\alpha}} \|\mathbf{F}\|_{L^\infty(S)}.
\end{aligned} \tag{3.51}$$

Finally, we get

$$\begin{aligned}
\|\widetilde{f}\|_{L^r(\widetilde{S})} &= \left(\int_{\widetilde{S}} (\det A_h)^{2r/n} f^r(Tx)\,dx\right)^{1/r}\\
&= \left(\int_S (\det A_h)^{(2r/n)-1} f^r(y)\,dy\right)^{1/r}\\
&\le \left(\int_S (Ch^{n/2})^{(2r/n)-1} f^r(y)\,dy\right)^{1/r}\\
&= C(n, \lambda, \Lambda)^{2/n-1/r} h^{1-n/2r} \|f\|_{L^r(S)}.
\end{aligned} \tag{3.52}$$

As $\partial \widetilde{S} = T^{-1}(\partial S)$, $\widetilde{u} = 0$ on $\partial \widetilde{S}$. Therefore, we may apply Lemma 3.4 to $\widetilde{u}$ and combine it with (3.51) and (3.52) to get

$$\begin{aligned} \|u\|_{L^\infty(S)} &= \|\widetilde{u}\|_{L^\infty(\widetilde{S})} \\ &\leq \widetilde{C}\left(\|\widetilde{\mathbf{F}}\|_{L^\infty(\widetilde{S})} + \|\widetilde{f}\|_{L^r(\widetilde{S})}\right) \\ &\leq \widetilde{C}\left(C(n,\lambda,\Lambda)h^{\frac{\alpha}{1+\alpha}}\|\mathbf{F}\|_{L^\infty(S)} + C(n,\lambda,\Lambda,r)h^{1-n/2r}\|f\|_{L^r(S)}\right), \end{aligned} \tag{3.53}$$

where

$$\widetilde{C} = C(n,\lambda,\Lambda,r,\varepsilon^*, \|\widetilde{\mathbf{b}}\|_{L^\infty(\widetilde{S})}, \|\widetilde{\mathbf{B}}\|_{L^\infty(\widetilde{S})}, \|D^2\widetilde{\varphi}\|_{L^{1+\varepsilon^*}(\widetilde{S})}). \tag{3.54}$$

As $S_\varphi(x_0, h)$ is contained in a normalized section, we have

$$h \leq C(n,\lambda,\Lambda). \tag{3.55}$$

Therefore, we have

$$\widetilde{C} \leq C(n,\lambda,\Lambda,r,\varepsilon^*, \|\mathbf{b}\|_{L^\infty(S)}, \|\mathbf{B}\|_{L^\infty(S)}). \tag{3.56}$$

Furthermore, the L^∞ norms of $\widetilde{\mathbf{b}}$, $\widetilde{\mathbf{B}}$ are under control by (3.50) and (3.51). Finally, by the $W^{2,1+\varepsilon}$ estimate in Theorem 2.7, we have

$$\|D^2\widetilde{\varphi}\|_{L^{1+\varepsilon^*}(\widetilde{S})} \leq C(n,\lambda,\Lambda).$$

Combining (3.53), (3.54), and (3.55), we have

$$\|u\|_{L^\infty(S)} \leq \widetilde{C}C(n,\lambda,\Lambda,r,\alpha)\left(\|\mathbf{F}\|_{L^\infty(S)} + \|f\|_{L^r(S)}\right)h^{\gamma(n,r,\alpha)}, \tag{3.57}$$

where

$$\gamma = \min\left\{1 - \frac{n}{2r}, \frac{\alpha}{1+\alpha}\right\}.$$

As $\alpha = \alpha(n,\lambda,\Lambda)$, the conclusion of the lemma follows from (3.56) and (3.57). □

4 Harnack inequality

In this section, we use the global estimate in Proposition 3.5 to prove the Harnack inequality, Theorem 1.1. We begin by expressing an arbitrary solution of (1.1) as the sum of solutions of a homogeneous equation and an inhomogeneous equation with zero boundary data. The inhomogeneous part can be bounded using Proposition 3.5, while the homogeneous part can be bounded using the Harnack inequality in Theorem 2.9. Combining these estimates yields the Harnack inequality in normalized sections, Proposition 4.1. Rescaling Proposition 4.1 then gives the desired Harnack inequality in Theorem 1.1.

We will first prove the following proposition.

Proposition 4.1 *(Harnack inequality in normalized section) Let $\varphi \in C^3(\Omega)$ be a convex function satisfying (1.3). Suppose that $\mathbf{F}, \mathbf{B}, \mathbf{b} \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \cap W^{1,n}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$, $f \in L^n_{\mathrm{loc}}(\Omega)$, $\operatorname{div}\mathbf{B} \leq 0$, and $n/2 < r \leq n$. Suppose $S_\varphi(x, h_0)$ is a normalized section contained in Ω, and $h \leq h_0/2$. Assume that $u \in W^{2,n}(S_\varphi(x,h))$ is a nonnegative solution to (1.1) in $S_\varphi(x,h)$. Further assume that*

1. *either $n = 2$, or*

2. $n \geq 3$ *and* $\varepsilon^*(n, \lambda, \Lambda) + 1 > \frac{n}{2}$, *where* ε^* *is from Theorem 2.7.*

Then,

$$\sup_{S_\varphi(x,h/2)} u \leq C\left((\|\mathbf{F}\|_{L^\infty(S_\varphi(x,h))} + \|f\|_{L^r(S_\varphi(x,h))})h^\gamma + \inf_{S_\varphi(x,h/2)} u \right),$$

where

$$\begin{aligned} &\gamma = \gamma(n, \lambda, \Lambda, r) > 0, \quad \textit{and} \\ &C = C(n, \lambda, \Lambda, r, \varepsilon^*, \|\mathbf{b}\|_{L^\infty(S_\varphi(x,h))}, \|\operatorname{div} \mathbf{B}\|_{L^n(S_\varphi(x,h))}, \|\mathbf{B}\|_{L^\infty(S_\varphi(x,h))}) > 0. \end{aligned}$$

Proof By [20, Theorem 9.15], we can find a solution $u_0 \in W^{2,n}(S)$ to

$$\begin{cases} -\operatorname{div}(\Phi D u_0 + u_0 \mathbf{B}) + \mathbf{b} \cdot D u_0 = f - \operatorname{div} \mathbf{F} & \text{in } S := S_\varphi(x, h), \\ u_0 = 0 & \text{on } \partial S. \end{cases}$$

Then $v = u - u_0$ satisfies $v \geq 0$ on ∂S, and is a solution to

$$-\operatorname{div}(\Phi D v + v\mathbf{B}) + \mathbf{b} \cdot D v = 0 \quad \text{in } S.$$

Observing that the equation above can be written as

$$-\Phi^{ij} D_{ij} v + (\mathbf{b} - \mathbf{B}) \cdot D v - (\operatorname{div} \mathbf{B}) v = 0$$

and $\operatorname{div} \mathbf{B} \leq 0$, we have $v \geq 0$ in S by the maximum principle [20, Theorem 9.1]. As $S_\varphi(x, 2h)$ is contained in a normalized section, $S_\varphi(x, 2h) \subset B_n$ and $h \leq C(n, \lambda, \Lambda)$ by Lemma 2.8. Therefore, we can apply the Harnack inequality in Theorem 2.9 to get

$$\sup_{S_\varphi(x,h/2)} v \leq C_1 \inf_{S_\varphi(x,h/2)} v, \tag{4.1}$$

where

$$C_1 = C_1(n, \lambda, \Lambda, \|\mathbf{b}\|_{L^\infty(S)}, \|\mathbf{B}\|_{L^\infty(S)}, \|\operatorname{div} \mathbf{B}\|_{L^n(S)}).$$

By applying the global estimate in Proposition 3.5 to u_0, we obtain

$$\sup_S |u_0| \leq C_2(\|\mathbf{F}\|_{L^\infty(S)} + \|f\|_{L^r(S)})h^\gamma, \tag{4.2}$$

where

$$\begin{aligned} C_2 &= C_2(n, \lambda, \Lambda, r, \varepsilon^*, \|\mathbf{b}\|_{L^\infty(S)}, \|\mathbf{B}\|_{L^\infty(S)}), \quad \text{and} \\ \gamma &= \gamma(n, \lambda, \Lambda, r) > 0. \end{aligned}$$

As $v = u - u_0$, combining (4.1) and (4.2) completes the proof. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1 We prove the theorem by using the rescaling scheme in the proof of Proposition 3.5. Using John's lemma, we find an affine transformation

$$Tx = A_h x + b_h$$

such that

$$TB_1 \subset S := S_\varphi(x_0, h) \subset TB_n, \tag{4.3}$$

Using the transformation T, we define the rescaled functions as in (3.45), (3.46).

We start by estimating the matrix A_h. First, from (4.3) and Lemma 2.8, we obtain the following bounds on $\det A_h$:

$$
\begin{aligned}
|\det A_h| &= \frac{|TB_1|}{|B_1|} \le \frac{|S_\varphi(x_0,h)|}{|B_1|} \le C_1(n,\lambda,\Lambda)h^{n/2} =: C_3(n,\lambda,\Lambda,h), \quad \text{and} \\
|\det A_h| &= \frac{|TB_n|}{|B_n|} \ge \frac{|S_\varphi(x_0,h)|}{|B_n|} \ge c_1(n,\lambda,\Lambda)h^{n/2} =: c_4(n,\lambda,\Lambda,h).
\end{aligned} \tag{4.4}
$$

We also have (see [31, (5.6)])

$$
\left\| A_h^{-1} \right\| \le C_2 = \widetilde{C}(n,\lambda,\Lambda,\operatorname{diam}(S_\varphi(x_0,2h)))h^{-n/2}. \tag{4.5}
$$

Recall that from (3.46),

$$
\widetilde{S} := T^{-1}(S_\varphi(x_0,h)) = S_{\widetilde{\varphi}}(y_0, (\det A_h)^{-2/n}h).
$$

We now estimate the rescaled functions. From (3.45), (4.5) and (4.4), we have

$$
\begin{aligned}
\|\widetilde{\mathbf{b}}\|_{L^\infty(\widetilde{S})} &\le C_3^{2/n} C_2 \|\mathbf{b}\|_{L^\infty(S)}, \\
\|\widetilde{\mathbf{B}}\|_{L^\infty(\widetilde{S})} &\le C_3^{2/n} C_2 \|\mathbf{B}\|_{L^\infty(S)}, \\
\|\widetilde{\mathbf{F}}\|_{L^\infty(\widetilde{S})} &\le C_3^{2/n} C_2 \|\mathbf{F}\|_{L^\infty(S)}, \quad \text{and} \\
\|\widetilde{f}\|_{L^r(\widetilde{S})} &= \left(\int_S (\det A_h)^{(2r/n)-1} f^r(y)\,dy \right)^{1/r} \\
&\le \left(\int_S C_3^{(2r/n)-1} f^r(y)\,dy \right)^{1/r} \\
&= C_3^{2/n-1/r} \|f\|_{L^r(S)}.
\end{aligned} \tag{4.6}
$$

Also, as

$$
\operatorname{div} \widetilde{\mathbf{B}}(x) = (\det A_h)^{2/n} \operatorname{div} \mathbf{B}(Tx) \le 0, \tag{4.7}
$$

we have

$$
\begin{aligned}
\|\operatorname{div} \widetilde{\mathbf{B}}\|_{L^n(\widetilde{S})} &= \left(\int_{\widetilde{S}} (\det A_h)^2 [(\operatorname{div} \mathbf{B})(Tx)]^n\,dx \right)^{1/n} \\
&= \left(\int_S (\det A_h) [(\operatorname{div} \mathbf{B})(y)]^n\,dy \right)^{1/n} \\
&\le \left(\int_S C_3 [(\operatorname{div} \mathbf{B})(y)]^n\,dy \right)^{1/n} \\
&= C_3^{1/n} \|\operatorname{div} \mathbf{B}\|_{L^n(S)}.
\end{aligned} \tag{4.8}
$$

For $t \le h/2$, setting

$$
\widetilde{t} = (\det A_h)^{-2/n} t \le (\det A_h)^{-2/n} h/2
$$

gives

$$
S_{\widetilde{\varphi}}(y_0, \widetilde{t}) = T^{-1} S_\varphi(x_0, t). \tag{4.9}
$$

Then, $\widetilde{u}$ is a solution to the rescaled equation (3.47) in $S_{\widetilde{\varphi}}(y_0, 2\widetilde{t})$. Applying Proposition 4.1 to $\widetilde{u}$, we get

$$\sup_{S_{\widetilde{\varphi}}(y_0,\widetilde{t})} \widetilde{u} \le C_5 \left\{ \left(\|\widetilde{\mathbf{F}}\|_{L^\infty(\widetilde{S})} + \|\widetilde{f}\|_{L^r(\widetilde{S})} \right) \widetilde{t}^\gamma + \inf_{S_{\widetilde{\varphi}}(y_0,\widetilde{t})} \widetilde{u} \right\}. \tag{4.10}$$

Here, the constants C_5 and γ come from Proposition 4.1:

$$\gamma = \gamma(n, \lambda, \Lambda, r) > 0, \quad \text{and}$$
$$C_5 = C_5(n, \lambda, \Lambda, r, \varepsilon^*, \|\widetilde{\mathbf{b}}\|_{L^\infty(\widetilde{S})}, \|\operatorname{div} \widetilde{\mathbf{B}}\|_{L^n(\widetilde{S})}, \|\widetilde{\mathbf{B}}\|_{L^\infty(\widetilde{S})}) > 0.$$

Furthermore, the norms $\|\widetilde{\mathbf{b}}\|_{L^\infty(\widetilde{S})}$, $\|\operatorname{div} \widetilde{\mathbf{B}}\|_{L^n(\widetilde{S})}$, $\|\widetilde{\mathbf{B}}\|_{L^\infty(\widetilde{S})}$, $\|\widetilde{\mathbf{F}}\|_{L^\infty(\widetilde{S})}$, and $\|\widetilde{f}\|_{L^r(\widetilde{S})}$ of the rescaled functions are under control by (4.6) and (4.8). Finally, $\widetilde{t}$ is controlled by t through

$$\widetilde{t} \le c_4^{-2/n} t. \tag{4.11}$$

Therefore, putting (4.6), (4.8), (4.9) and (4.11) together, we obtain the conclusion of the theorem from (4.10). □

5 Interior estimates

In this section, we prove the interior estimate for solutions to (1.1) in Lemma 5.2. This estimate will be used in the proofs of the Hölder estimates in Corollary 1.2 and Theorem 1.3 in Sect. 6.

We begin by defining suitable test functions and then applying Moser iteration. This yields an estimate in Lemma 5.1 for the L^∞ norm of solutions u to (1.1), involving its L^{q^*} norm in a larger section, where q^* is a finite number. Next, using a dilation argument from Le [31, Theorem 15.4] and rescaling, we obtain the interior estimate in Lemma 5.2.

We will first prove the following lemma.

Lemma 5.1 *(Interior estimate in normalized section) Let $\varphi \in C^3(\Omega)$ be a convex function satisfying (1.3). Suppose $\mathbf{F}, \mathbf{b}, \mathbf{B} \in W^{1,n}_{\text{loc}}(\Omega; \mathbb{R}^n) \cap L^\infty_{\text{loc}}(\Omega; \mathbb{R}^n)$, $f \in L^n_{\text{loc}}(\Omega)$, and $n/2 < r \le n$. Assume that $S_\varphi(x_0, 2t) \Subset \Omega$, and $S_\varphi(x_0, t)$ is a normalized section. Assume that $u \in W^{1,2}(S_\varphi(x_0, t))$ is a nonnegative solution to (1.1) in $S_\varphi(x_0, t)$. Further assume that*

1. *either $n = 2$, or*
2. *$n \ge 3$ and $\varepsilon^*(n, \lambda, \Lambda) + 1 > \frac{n(n-1)}{2}$ where ε^* is from Theorem 2.7.*

Then,

$$\sup_{S_\varphi(x_0,t/2)} u \le C(\|u\|_{L^{q^*}(S_\varphi(x_0,t))} + \|\mathbf{F}\|_{L^\infty(S_\varphi(x_0,t))} + \|f\|_{L^r(S_\varphi(x_0,t))}),$$

where

$$C = C(n, \lambda, \Lambda, r, \varepsilon^*, \|\mathbf{b}\|_{L^\infty(S_\varphi(x_0,t))}, \|\mathbf{B}\|_{L^\infty(S_\varphi(x_0,t))}), \quad \text{and} \quad q^* = q^*(\varepsilon^*, n, r).$$

Proof We argue as in Le [31, pp.515–517]. Let $\overline{u} = u + k$, where

$$k = \|\mathbf{F}\|_{L^\infty(S_\varphi(x_0,t))} + \|f\|_{L^r(S_\varphi(x_0,t))}.$$

For $\eta \in C^1_c(S_\varphi(x_0, t))$ to be determined later and $\beta \ge 0$, we use $v = \eta^2 \overline{u}^{\beta+1} \in C^1_c(S_\varphi(x_0, t))$ as a test function in (1.1) and extend it to be zero outside $S_\varphi(x_0, t)$ to obtain

$$\int_\Omega \Phi Du \cdot Dv\, dx + \int_\Omega u\mathbf{B} \cdot Dv\, dx + \int_\Omega v\mathbf{b} \cdot Du\, dx = \int_\Omega \mathbf{F} \cdot Dv\, dx + \int_\Omega f v. \tag{5.1}$$

Because

$$Dv = (\beta+1)\eta^2\overline{u}^\beta D\overline{u} + 2\eta\overline{u}^{\beta+1}D\eta \quad \text{and } D\overline{u} = Du,$$

the terms in (5.1) become

$$\begin{aligned}
\int_\Omega \Phi Du \cdot Dv\, dx &= (\beta+1)\int_\Omega \eta^2\overline{u}^\beta \Phi D\overline{u}\cdot D\overline{u}\, dx + 2\int_\Omega \eta\overline{u}^{\beta+1}\Phi D\overline{u}\cdot D\eta\, dx,\\
\int_\Omega u\mathbf{B}\cdot Dv\, dx &= (\beta+1)\int_\Omega \eta^2 u\overline{u}^\beta \mathbf{B}\cdot D\overline{u}\, dx + 2\int_\Omega \eta u\overline{u}^{\beta+1}\mathbf{B}\cdot D\eta\, dx,\\
\int_\Omega v\mathbf{b}\cdot Du\, dx &= \int_\Omega \eta^2\overline{u}^{\beta+1}\mathbf{b}\cdot D\overline{u}\, dx,\\
\int_\Omega \mathbf{F}\cdot Dv\, dx &= (\beta+1)\int_\Omega \eta^2\overline{u}^\beta \mathbf{F}\cdot D\overline{u}\, dx + 2\int_\Omega \eta\overline{u}^{\beta+1}\mathbf{F}\cdot D\eta\, dx, \quad \text{and}\\
\int_\Omega fv\, dx &= \int_\Omega \eta^2\overline{u}^{\beta+1} f\, dx.
\end{aligned} \tag{5.2}$$

We now estimate these terms. By the Cauchy-Schwarz inequality, we have

$$-2\int_\Omega \eta\overline{u}^{\beta+1}\Phi D\overline{u}\cdot D\eta\, dx \le \frac{1}{8}\int_\Omega \eta^2\overline{u}^\beta\Phi D\overline{u}\cdot D\overline{u}\, dx + 8\int_\Omega \overline{u}^{\beta+2}\Phi D\eta\cdot D\eta\, dx. \tag{5.3}$$

By the Cauchy-Schwarz inequality and (3.3), we get

$$\begin{aligned}
&-(\beta+1)\int_\Omega \eta^2 u\overline{u}^\beta\mathbf{B}\cdot D\overline{u}\, dx\\
&\le (\beta+1)\int_\Omega \eta^2\overline{u}^{\beta+1}|\mathbf{B}||D\overline{u}|\, dx\\
&\le \frac{\beta+1}{8}\int_\Omega \eta^2\overline{u}^\beta\Phi D\overline{u}\cdot D\overline{u}\, dx + 2(\beta+1)\int_\Omega \eta^2\overline{u}^{\beta+2}\frac{\Delta\varphi}{\lambda}|\mathbf{B}|^2\, dx.
\end{aligned} \tag{5.4}$$

Similarly,

$$\begin{aligned}
-2\int_\Omega \eta u\overline{u}^{\beta+1}\mathbf{B}\cdot D\eta\, dx &\le 2\int_\Omega \eta\overline{u}^{\beta+2}|\mathbf{B}||D\eta|\, dx\\
&\le \int_\Omega \overline{u}^{\beta+2}\Phi D\eta\cdot D\eta\, dx + \int_\Omega \eta^2\overline{u}^{\beta+2}\frac{\Delta\varphi}{\lambda}|\mathbf{B}|^2\, dx,
\end{aligned} \tag{5.5}$$

and

$$-\int_\Omega \eta^2\overline{u}^{\beta+1}\mathbf{b}\cdot D\overline{u}\, dx \le \frac{1}{8}\int_\Omega \eta^2\overline{u}^\beta\Phi D\overline{u}\cdot D\overline{u}\, dx + 2\int_\Omega \eta^2\overline{u}^{\beta+2}\frac{\Delta\varphi}{\lambda}|\mathbf{b}|^2\, dx. \tag{5.6}$$

As in (5.3) and using $\overline{u} \ge |\mathbf{F}|$ in $S_\varphi(x_0,t)$, we have

$$\begin{aligned}
&(\beta+1)\int_\Omega \eta^2\overline{u}^\beta\mathbf{F}\cdot D\overline{u}\, dx\\
&\le (\beta+1)\int_\Omega \left(\eta^2\overline{u}^\beta\Phi D\overline{u}\cdot D\overline{u}\right)^{1/2}\left(\eta^2\overline{u}^\beta\frac{\Delta\varphi}{\lambda}|\mathbf{F}|^2\right)^{1/2} dx\\
&\le \frac{\beta+1}{8}\int_\Omega \eta^2\overline{u}^\beta\Phi D\overline{u}\cdot D\overline{u}\, dx + 2(\beta+1)\int_\Omega \eta^2\overline{u}^\beta\frac{\Delta\varphi}{\lambda}|\mathbf{F}|^2\, dx\\
&\le \frac{\beta+1}{8}\int_\Omega \eta^2\overline{u}^\beta\Phi D\overline{u}\cdot D\overline{u}\, dx + 2(\beta+1)\int_\Omega \eta^2\overline{u}^{\beta+2}\frac{\Delta\varphi}{\lambda}\, dx,
\end{aligned} \tag{5.7}$$

and

$$
\begin{aligned}
2\int_{\Omega} \eta \overline{u}^{\beta+1}\mathbf{F}\cdot D\eta\,dx &\le 2\int_{\Omega}\left(\overline{u}^{\beta+2}\Phi D\eta\cdot D\eta\right)^{1/2}\left(\eta^2\overline{u}^{\beta}\frac{\Delta\varphi}{\lambda}|\mathbf{F}|^2\right)^{1/2}\,dx \\
&\le \int_{\Omega}\overline{u}^{\beta+2}\Phi D\eta\cdot D\eta\,dx + \int_{\Omega}\eta^2\overline{u}^{\beta}\frac{\Delta\varphi}{\lambda}|\mathbf{F}|^2\,dx \\
&\le \int_{\Omega}\overline{u}^{\beta+2}\Phi D\eta\cdot D\eta\,dx + \int_{\Omega}\eta^2\overline{u}^{\beta+2}\frac{\Delta\varphi}{\lambda}\,dx.
\end{aligned}
\tag{5.8}
$$

Finally,

$$
\int_{\Omega}\eta^2\overline{u}^{\beta+1}f\,dx \le \int_{\Omega}\eta^2\overline{u}^{\beta+2}\frac{|f|}{k}\,dx. \tag{5.9}
$$

Now we put (5.1)–(5.9) together. We use (5.2) to substitute the integrals in (5.1); then, we apply the estimates in (5.3)–(5.9). As $\beta \ge 0$ and each integral appearing on the right-hand sides of (5.3)–(5.9) is nonnegative, we get

$$
\begin{aligned}
&\frac{1}{2}(\frac{\beta}{2}+1)\int_{\Omega}\eta^2\overline{u}^{\beta}\Phi D\overline{u}\cdot D\overline{u}\,dx \\
&\le 10\left(\int_{\Omega}\overline{u}^{\beta+2}\Phi D\eta\cdot D\eta\,dx + \frac{\beta+2}{2}\int_{\Omega}\eta^2\overline{u}^{\beta+2}\left\{\frac{\Delta\varphi}{\lambda}(1+|\mathbf{b}|^2+|\mathbf{B}|^2)+\frac{|f|}{k}\right\}\,dx\right).
\end{aligned}
\tag{5.10}
$$

Because

$$
D(\overline{u}^{\beta/2+1}\eta) = \left(\frac{\beta}{2}+1\right)\overline{u}^{\beta/2}\eta D\overline{u} + \overline{u}^{\beta/2+1}D\eta,
$$

we have

$$
\Phi D(\overline{u}^{\beta/2+1}\eta)\cdot D(\overline{u}^{\beta/2+1}\eta) \le 2\left[\left(\frac{\beta}{2}+1\right)^2\overline{u}^{\beta}\eta^2\Phi D\overline{u}\cdot D\overline{u} + \overline{u}^{\beta+2}\Phi D\eta\cdot D\eta\right].
$$

Therefore, (5.10) implies that

$$
\begin{aligned}
\int_{\Omega}\Phi D(\overline{u}^{\beta/2+1}\eta)\cdot D(\overline{u}^{\beta/2+1}\eta)\,dx &\le 128\left(\frac{\beta}{2}+1\right)^2\left[\int_{\Omega}\overline{u}^{\beta+2}\Phi D\eta\cdot D\eta\,dx\right. \\
&\left.+\int_{\Omega}\eta^2\overline{u}^{\beta+2}\left\{\frac{\Delta\varphi}{\lambda}(1+|\mathbf{b}|^2+|\mathbf{B}|^2)+\frac{|f|}{k}\right\}\,dx\right].
\end{aligned}
\tag{5.11}
$$

Letting

$$
S_a := S_{\varphi}(x_0, a),
$$

we have, from the Alexandrov Maximum Principle [31, Theorem 3.12] (also see [31, (15.16)]),

$$
\operatorname{dist}(S_{\overline{r}}, \partial S_R) \ge c(n,\lambda,\Lambda)(R-\overline{r})^n \quad \text{for } 0 < \overline{r} < R \le t.
$$

Hence, we may choose η supported on S_R so that $0 \le \eta \le 1$, $\eta \equiv 1$ in $S_{\overline{r}}$, and

$$
|D\eta| \le C_0(n,\lambda,\Lambda)(R-\overline{r})^{-n}. \tag{5.12}
$$

We set

$$q=\min\left\{\frac{1+\varepsilon^*}{n-1},r\right\}>\frac{n}{2},$$

and define $\widehat{q},\widehat{n}$ using (3.10) and (3.13). That is,

$$\widehat{q}:=\frac{2q}{q-1},\quad \text{and } \widehat{n}:=\begin{cases}\frac{2n}{n-2} & \text{if } n\geq 3,\\ 2\widehat{q} & \text{if } n=2.\end{cases}$$

Then, by the Monge–Ampère Sobolev inequality, Theorem 2.4, we have

$$\begin{aligned}\int_\Omega \Phi D(\overline{u}^{\beta/2+1}\eta)\cdot D(\overline{u}^{\beta/2+1}\eta)\,dx &= \int_{S_R}\Phi D(\overline{u}^{\beta/2+1}\eta)\cdot D(\overline{u}^{\beta/2+1}\eta)\,dx\\ &\geq c_1(n,\lambda,\Lambda,\widehat{q})\left\|\overline{u}^{\beta/2+1}\eta\right\|^2_{L^{\widehat{n}}(S_R)}\\ &\geq c_1\left\|\overline{u}^{\beta/2+1}\right\|^2_{L^{\widehat{n}}(S_{\overline{r}})}.\end{aligned}\tag{5.13}$$

Because $D^2\varphi>0$, all of its eigenvalues are smaller than $\Delta\varphi$. Hence,

$$\Phi=(\det D^2\varphi)(D^2\varphi)^{-1}\leq(\Delta\varphi)^{n-1}I_n.$$

Therefore, we have, from (5.12),

$$\begin{aligned}\int_\Omega \overline{u}^{\beta+2}\Phi D\eta\cdot D\eta\,dx &\leq \int_{S_R}\overline{u}^{\beta+2}(\Delta\varphi)^{n-1}|D\eta|^2\,dx\\ &\leq C_0^2(R-\overline{r})^{-2n}\int_{S_R}\overline{u}^{\beta+2}(\Delta\varphi)^{n-1}\,dx.\end{aligned}\tag{5.14}$$

We also have

$$\begin{aligned}&\int_\Omega \eta^2\overline{u}^{\beta+2}\left\{\frac{\Delta\varphi}{\lambda}(1+|\mathbf{b}|^2+|\mathbf{B}|^2)+\frac{|f|}{k}\right\}dx\\ &\leq \int_{S_R}\overline{u}^{\beta+2}\left\{\frac{\Delta\varphi}{\lambda}(1+|\mathbf{b}|^2+|\mathbf{B}|^2)+\frac{|f|}{k}\right\}dx\\ &\leq C_2(n,\lambda,\Lambda)(R-\overline{r})^{-2n}\int_{S_R}\overline{u}^{\beta+2}\left\{\frac{\Delta\varphi}{\lambda}(1+|\mathbf{b}|^2+|\mathbf{B}|^2)+\frac{|f|}{k}\right\}dx.\end{aligned}\tag{5.15}$$

If we define

$$h:=\frac{\Delta\varphi}{\lambda}(1+|\mathbf{b}|^2+|\mathbf{B}|^2)+\frac{|f|}{k}+(\Delta\varphi)^{n-1},$$

then $h\in L^q(S_t)$. From (5.14), (5.15), and the Hölder inequality, the right-hand side of (5.11) is bounded by

$$\begin{aligned}\text{RHS (5.11)} &\leq 128(C_0^2+C_2)(\frac{\beta}{2}+1)^2(R-\overline{r})^{-2n}\int_{S_R}\overline{u}^{\beta+2}h\,dx\\ &\leq C_3(n,\lambda,\Lambda)(\frac{\beta}{2}+1)^2(R-\overline{r})^{-2n}\left\|\overline{u}^{\beta/2+1}\right\|^2_{L^{\widehat{q}}(S_R)}\|h\|_{L^q(S_t)}.\end{aligned}\tag{5.16}$$

Combining (5.11), (5.13), and (5.16) yields

$$\left\|\overline{u}^{\beta/2+1}\right\|^2_{L^{\widehat{n}}(S_{\overline{r}})}\leq C_4(n,\lambda,\Lambda,\widehat{q})\|h\|_{L^q(S_t)}(R-\overline{r})^{-2n}(\frac{\beta}{2}+1)^2\left\|\overline{u}^{\beta/2+1}\right\|^2_{L^{\widehat{q}}(S_R)}.\tag{5.17}$$

As $q > n/2, \widehat{n} > \widehat{q}$ and we may set

$$\chi := \frac{\widehat{n}}{\widehat{q}} > 1, \quad \text{and } \gamma := \widehat{q}(\frac{\beta}{2} + 1).$$

Then, (5.17) becomes

$$\|\overline{u}\|_{L^{\gamma\chi}(S_{\overline{r}})} \leq \left(C_5(n, \lambda, \Lambda, \widehat{q}) \|h\|_{L^q(S_t)} (R - \overline{r})^{-2n}\gamma^2\right)^{\frac{q}{q-1}\frac{1}{\gamma}} \|\overline{u}\|_{L^\gamma(S_R)}. \tag{5.18}$$

Define for each integer $j \geq 0$

$$r_j = \frac{t}{2} + \frac{t}{2^{j+1}}, \quad \text{and } \gamma_j = \chi^j \widehat{q}.$$

Setting $R = r_j, \overline{r} = r_{j+1}$, and $\gamma = \gamma_j$ in (5.18), we get

$$\|\overline{u}\|_{L^{\chi^{j+1}\widehat{q}}(S_{r_{j+1}})} \leq \left(2C_5\widehat{q}^2 \|h\|_{L^q(S_t)} t^{-2n} 2^{2n(j+2)} \chi^{2j}\right)^{\chi^{-j}/2} \|\overline{u}\|_{L^{\chi^j}(S_{r_j})}. \tag{5.19}$$

Iterating (5.19) yields

$$\|\overline{u}\|_{L^\infty(S_{t/2})} \leq \left(2C_5\widehat{q}^2 \|h\|_{L^q(S_t)} t^{-2n}\right)^{\sum_{j\geq 0}\chi^{-j}/2} 2^{\sum_{j\geq 0} n(j+2)\chi^{-j}} \chi^{\sum_{j\geq 0} j\chi^{-j}} \|\overline{u}\|_{L^{\widehat{q}}(S_t)}. \tag{5.20}$$

As $S_\varphi(x_0, t)$ is normalized, we have from Lemma 2.8,

$$t^{-1} \leq C_6(n, \lambda, \Lambda). \tag{5.21}$$

Finally, the $W^{2,1+\varepsilon}$ estimate in Theorem 2.7 implies

$$\begin{aligned}
\|h\|_{L^q(S_t)} &\leq C_7(n) \frac{1 + \|\mathbf{b}\|^2_{L^\infty(S_t)} + \|\mathbf{B}\|^2_{L^\infty(S_t)}}{\lambda} \left\|D^2\varphi\right\|_{L^q(S_t)} \\
&\quad + \frac{\|f\|_{L^q(S_t)}}{k} + C_7(n) \left\|D^2\varphi\right\|^{n-1}_{L^{q(n-1)}(S_t)} \\
&\leq C_8(n, \varepsilon^*, r, \lambda, \Lambda, \|\mathbf{b}\|_{L^\infty(S_t)}, \|\mathbf{B}\|_{L^\infty(S_t)}, \left\|D^2\varphi\right\|_{L^{1+\varepsilon^*}(S_t)}) \\
&\leq C_9(n, \varepsilon^*, r, \lambda, \Lambda, \|\mathbf{b}\|_{L^\infty(S_t)}, \|\mathbf{B}\|_{L^\infty(S_t)}).
\end{aligned} \tag{5.22}$$

The conclusion of the lemma follows from (5.20)–(5.22). □

Now, we rescale (1.1) as we did in the proof of Proposition 3.5, and apply the result in Lemma 5.1. Using the estimates from the proof of Theorem 1.1, we then argue as in Le [31, Theorem 15.4] to obtain the following interior estimates in general sections.

Lemma 5.2 *(Interior estimate in general section) Let $\varphi \in C^3(\Omega)$ be a convex function satisfying (1.3). Suppose $\mathbf{F}, \mathbf{b}, \mathbf{B} \in W^{1,n}_{\text{loc}}(\Omega; \mathbb{R}^n) \cap L^\infty_{\text{loc}}(\Omega; \mathbb{R}^n)$, $f \in L^n_{\text{loc}}(\Omega)$, and $n/2 < r \leq n$. Assume that $S_\varphi(x_0, 2h) \Subset \Omega$ and $u \in W^{1,2}(S_\varphi(x_0, h))$ is a nonnegative solution to (1.1) in $S_\varphi(x_0, 2h)$. Further assume that*

1. *either $n = 2$, or*
2. *$n \geq 3$ and $\varepsilon^*(n, \lambda, \Lambda) + 1 > \frac{n(n-1)}{2}$ where ε^* is from Theorem 2.7.*

Then,

$$\sup_{S_\varphi(x_0, h/2)} u \leq C(h^{-\frac{n}{4}} \|u\|_{L^2(S_\varphi(x_0,h))} + h^{1-\frac{n}{2}} \|\mathbf{F}\|_{L^\infty(S_\varphi(x_0,h))} + h^{1-\frac{n}{2r}} \|f\|_{L^r(S_\varphi(x_0,h))}) \tag{5.23}$$

where

$$C = C(n, \lambda, \Lambda, r, \varepsilon^*, \|\mathbf{b}\|_{L^\infty(S_\varphi(x_0,h))}, \|\mathbf{B}\|_{L^\infty(S_\varphi(x_0,h))}, h, \mathrm{diam}(S_\varphi(x, 2h))) > 0.$$

Proof We rescale $S = S_\varphi(x_0, h)$ as in the proof of Theorem 1.1, so that $B_1 \subset T^{-1} S_\varphi(x_0, h) \subset B_n$. We will use $\widetilde{C}$ and the numbered constants C_n to denote the same constants from the proof of Theorem 1.1 throughout the proof of this lemma.

For $\widetilde{h} := (\det A_h)^{-2/n} h$, we have the rescaled equation (3.47) in $\widetilde{S} = S_{\widetilde{\varphi}}(y_0, \widetilde{h})$. Applying Lemma 5.1 to $\widetilde{u}$, we get

$$\sup_{S_{\widetilde{\varphi}}(y_0,\widetilde{h}/2)} \widetilde{u} \le D_1(\|\widetilde{u}\|_{L^{q^*}(S_{\widetilde{\varphi}}(y_0,\widetilde{h}))} + \|\widetilde{\mathbf{F}}\|_{L^\infty(S_{\widetilde{\varphi}}(y_0,\widetilde{h}))} + \|\widetilde{f}\|_{L^r(S_{\widetilde{\varphi}}(y_0,\widetilde{h}))}), \tag{5.24}$$

where $D_1 > 0$ depends on $n, \lambda, \Lambda, r, \varepsilon^*$, $\|\widetilde{\mathbf{b}}\|_{L^\infty(\widetilde{S})}$, and $\|\widetilde{\mathbf{B}}\|_{L^\infty(\widetilde{S})}$.

Using the expression for C_2 in (4.5) and C_3 in (4.4), we use the estimates (4.6) to estimate the norms of the rescaled functions:

$$\begin{aligned}
\|\widetilde{\mathbf{b}}\|_{L^\infty(S_{\widetilde{\varphi}}(y_0,\widetilde{h}))} &\le \left(C_1 h^{n/2}\right)^{2/n} (\widetilde{C} h^{-n/2}) \|\mathbf{b}\|_{L^\infty(S_\varphi(x_0,h))} \\
&= C_1^{2/n} \widetilde{C} h^{1-n/2} \|\mathbf{b}\|_{L^\infty(S_\varphi(x_0,h))}, \\
\|\widetilde{\mathbf{B}}\|_{L^\infty(S_{\widetilde{\varphi}}(y_0,\widetilde{h}))} &\le C_1^{2/n} \widetilde{C} h^{1-n/2} \|\mathbf{B}\|_{L^\infty(S_\varphi(x_0,h))}, \\
\|\widetilde{\mathbf{F}}\|_{L^\infty(S_{\widetilde{\varphi}}(y_0,\widetilde{h}))} &\le C_1^{2/n} \widetilde{C} h^{1-n/2} \|\mathbf{F}\|_{L^\infty(S_\varphi(x_0,h))}, \quad \text{and} \\
\|\widetilde{f}\|_{L^r(S_{\widetilde{\varphi}}(y_0,\widetilde{h}))} &= (C_1 h^{n/2})^{2/n-1/r} \|f\|_{L^r(S_\varphi(x_0,h))} \\
&= C_1^{2/n-1/r} h^{1-n/2r} \|f\|_{L^r(S_\varphi(x_0,h))}.
\end{aligned} \tag{5.25}$$

We also have (see [31, Lemma 15.2(iii)])

$$\|\widetilde{u}\|_{L^{q^*}(S_{\widetilde{\varphi}}(y_0,\widetilde{h}))} \le D_2(n, \lambda, \Lambda, q^*) h^{-n/2q^*} \|u\|_{L^{q^*}(S_\varphi(x_0,h))} \tag{5.26}$$

for $q^* = q^*(\varepsilon^*, n, r)$, and

$$\sup_{S_{\widetilde{\varphi}}(y_0,\widetilde{h}/2)} \widetilde{u} = \sup_{S_\varphi(x_0,h/2)} u. \tag{5.27}$$

The L^∞ norms of $\widetilde{\mathbf{b}}$ and $\widetilde{\mathbf{B}}$ are under control from (5.25). Hence, from (5.24)–(5.27) we have,

$$\sup_{S_\varphi(x_0,h/2)} u \le D_3(h^{-\frac{n}{2q^*}} \|u\|_{L^{q^*}(S_\varphi(x_0,h))} + h^{1-\frac{n}{2}} \|\mathbf{F}\|_{L^\infty(S_\varphi(x_0,h))} + h^{1-\frac{n}{2r}} \|f\|_{L^r(S_\varphi(x_0,h))}), \tag{5.28}$$

where D_3 depends on $n, \lambda, \Lambda, r, \varepsilon^*$, $\|(\mathbf{b}, \mathbf{B})\|_{L^\infty(S_\varphi(x_0,h))}$, h, and $\mathrm{diam}(S_\varphi(x_0, 2h))$. We can now use (5.28) to argue as in Le [31, pp.519–521] (see also Han-Lin [21, pp.75–76]) to obtain (5.23). This gives the conclusion of the Lemma. □

Remark 5.3 In fact, following the arguments cited above, we can obtain (5.23) with the L^2 norm of u replaced by the L^p norm of u, for any $p > 0$.

6 Interior Hölder estimates

In this section, we prove the interior Hölder estimates in Corollary 1.2 and Theorem 1.3. We start by combining the Harnack inequality in Theorem 1.1 and the global estimate in Proposition 3.5 to prove Corollary 1.2.

Proof of Corollary 1.2 Let $\operatorname{osc}(g, E) := \sup_E g - \inf_E g$. It is sufficient (see [31, pp.523–524]) to prove the oscillation estimate

$$\operatorname{osc}(u, S_\varphi(x_0, h)) \le C_0 \left(\|u\|_{L^\infty(S_\varphi(x_0,h_0))} + \|\mathbf{F}\|_{L^\infty(S_\varphi(x_0,2h_0))} + \|f\|_{L^r(S_\varphi(x_0,2h_0))} \right) h^{\gamma_0} \tag{6.1}$$

for all $h \le h_0$, where the positive constants C_0 and γ_0 have the same dependency as C and γ stated in the Corollary.

As in Le [26, pp.284–285], we break up the solution $u = v + w$ in $S_\varphi(x_0, h)$, $h \le h_0$, where $v, w \in W^{2,n}(S_\varphi(x_0, h))$ are solutions to

$$\begin{cases} -\operatorname{div}(\Phi Dv + v\mathbf{B}) + \mathbf{b} \cdot Dv = f - \operatorname{div} \mathbf{F} & \text{in } S_\varphi(x_0, h), \\ v = 0 & \text{on } \partial S_\varphi(x_0, h), \end{cases}$$

and

$$\begin{cases} -\operatorname{div}(\Phi Dw + w\mathbf{B}) + \mathbf{b} \cdot Dw = 0 & \text{in } S_\varphi(x_0, h), \\ w = u & \text{on } \partial S_\varphi(x_0, h). \end{cases}$$

Such u and v exist as a consequence of [20, Theorem 9.15].

We now rescale $S_\varphi(x_0, 2h_0)$ as in the proof of Theorem 1.1, so that $B_1 \subset T^{-1} S_\varphi(x_0, 2h_0) \subset B_n$ for $Tx = A_{2h_0}x + b_{2h_0}$. We define the rescaled functions using (3.45), and set $\widetilde{v}(x) := v(Tx)$. Applying the global estimate in Proposition 3.5 to $\widetilde{v}$, we get

$$\|\widetilde{v}\|_{L^\infty(S_{\widetilde{\varphi}}(y_0,\widetilde{h}))} \le \widetilde{C}_1 \left(\|\widetilde{\mathbf{F}}\|_{L^\infty(S_{\widetilde{\varphi}}(y_0,\widetilde{h}))} + \|\widetilde{f}\|_{L^r(S_{\widetilde{\varphi}}(y_0,\widetilde{h}))} \right) \widetilde{h}^{\gamma_1}, \tag{6.2}$$

where

$$\begin{aligned} \widetilde{h} &:= (\det A_{2h_0})^{-2/n} h, \\ \widetilde{C}_1 &= \widetilde{C}_1 \left(n, \lambda, \Lambda, r, \varepsilon^*, \|\widetilde{\mathbf{B}}\|_{L^\infty(S_{\widetilde{\varphi}}(y_0,\widetilde{h}))}, \|\widetilde{\mathbf{b}}\|_{L^\infty(S_{\widetilde{\varphi}}(y_0,\widetilde{h}))} \right), \quad \text{and} \\ \gamma_1 &= \gamma_1(n, \lambda, \Lambda, r) > 0. \end{aligned}$$

The L^∞ norms of $\widetilde{\mathbf{F}}$, $\widetilde{\mathbf{b}}$, $\widetilde{\mathbf{B}}$, and the L^r norm of $\widetilde{f}$ are under control as in (5.25). Also, by (4.4),

$$\widetilde{h} \le C(n, \lambda, \Lambda, h_0) h.$$

Hence, from (6.2), we get

$$\operatorname{osc}(v, S_\varphi(x_0, h/2)) \le 2\,\|v\|_{L^\infty(S_\varphi(x_0,h))} \le C_1 (\|\mathbf{F}\|_{L^\infty(S_\varphi(x_0,2h_0))} + \|f\|_{L^r(S_\varphi(x_0,2h_0))}) h^{\gamma_1}, \tag{6.3}$$

where

$$C_1 = C_1(n, \lambda, \Lambda, r, \varepsilon^*, \|\mathbf{b}\|_{L^\infty(S_\varphi(x_0,2h_0))}, \|\mathbf{B}\|_{L^\infty(S_\varphi(x_0,2h_0))}, h_0, \operatorname{diam}(S_\varphi(x_0, 2h_0))).$$

We now estimate the oscillation of w. Define

$$M(t) := \sup_{S_\varphi(x_0,t)} w \quad \text{and}\ m(t) := \inf_{S_\varphi(x_0,t)} w,$$

and set

$$w_1(x) := w(x) - m(h) \quad \text{and}\ w_2(x) := M(h) - w(x).$$

Then, w_1 and w_2 are nonnegative solutions to

$$\begin{aligned} &-\operatorname{div}(\Phi D w_1 + w_1 \mathbf{B}) + \mathbf{b} \cdot D w_1 = m(h) \operatorname{div} \mathbf{B}, \quad \text{and} \\ &-\operatorname{div}(\Phi D w_2 + w_2 \mathbf{B}) + \mathbf{b} \cdot D w_2 = -M(h) \operatorname{div} \mathbf{B} \end{aligned}$$

in $S_\varphi(x_0, h)$. Therefore, applying the Harnack inequality in Theorem 1.1 to w_1, w_2 gives

$$\begin{aligned} &M(h/2) - m(h) \leq C_2(m(h/2) - m(h) + \|m(h) \operatorname{div} \mathbf{B}\|_{L^n(S_\varphi(x_0,h_0))}\, h^{\gamma_2}), \quad \text{and} \\ &M(h) - m(h/2) \leq C_2(M(h) - M(h/2) + \|M(h) \operatorname{div} \mathbf{B}\|_{L^n(S_\varphi(x_0,h_0))}\, h^{\gamma_2}), \end{aligned} \tag{6.4}$$

where

$$C_2 = C_2\left(n, \lambda, \Lambda, \varepsilon^*, \|(\mathbf{b}, \mathbf{B})\|_{L^\infty(S_\varphi(x_0,2h_0))}, \|\operatorname{div} \mathbf{B}\|_{L^n(S_\varphi(x_0,2h_0))}, h_0, \operatorname{diam}(S_\varphi(x_0, 2h_0))\right),$$

and

$\gamma_2 = \gamma_2(n, \lambda, \Lambda) > 0.$

Note that w satisfies a nondivergence form equation in $S_\varphi(x_0, h)$. That is,

$$-\Phi_{ij} D_{ij} w + (\mathbf{b} - \mathbf{B}) \cdot D w - (\operatorname{div} \mathbf{B}) w = 0.$$

As $\operatorname{div} \mathbf{B} \leq 0$, we may apply the maximum principle [20, Theorem 9.1] using the nondivergence form equation to conclude that w takes extreme values on ∂S. As $w = u$ on ∂S,

$$|M(h)|, |m(h)| \leq \|u\|_{L^\infty(S_\varphi(x_0,h))}\,.$$

Therefore, as $h \leq h_0$, we have

$$\begin{aligned} &\|m(h) \operatorname{div} \mathbf{B}\|_{L^n(S_\varphi(x_0,h_0))} + \|M(h) \operatorname{div} \mathbf{B}\|_{L^n(S_\varphi(x_0,h_0))} \\ &\quad \leq 2\, \|u\|_{L^\infty(S_\varphi(x_0,h_0))}\, \|\operatorname{div} \mathbf{B}\|_{L^n(S_\varphi(x_0,2h_0))}\,. \end{aligned}$$

Hence, adding the two inequalities in (6.4), we get

$$\begin{aligned} &(1 + C_2)(M(h/2) - m(h/2)) \\ &\quad \leq (C_2 - 1)(M(h) - m(h)) + 2C_2\, \|\operatorname{div} \mathbf{B}\|_{L^n(S_\varphi(x_0,2h_0))}\, \|u\|_{L^\infty(S_\varphi(x_0,h_0))}\, h^{\gamma_2}. \end{aligned}$$

Replacing C_2 by $C_2 + 2$, we may assume $C_2 > 1$. Setting $\beta := \frac{C_2-1}{C_2+1} \in (0, 1)$ and

$$C_3 := \frac{2C_2\, \|\operatorname{div} \mathbf{B}\|_{L^n(S_\varphi(x_0,2h_0))}}{1 + C_2},$$

we have

$$\operatorname{osc}(w, S_\varphi(x_0, h/2)) \leq \beta \operatorname{osc}(w, S_\varphi(x_0, h)) + C_3\, \|u\|_{L^\infty(S_\varphi(x_0,h_0))}\, h^{\gamma_2}. \tag{6.5}$$

From the maximum principle, we also have

$$\operatorname{osc}(w, S_\varphi(x_0, h)) = \operatorname{osc}(w, \partial S_\varphi(x_0, h)) = \operatorname{osc}(u, \partial S_\varphi(x_0, h)) \leq \operatorname{osc}(u, S_\varphi(x_0, h)). \tag{6.6}$$

Recalling $u = v + w$, from (6.3), (6.5), and (6.6) we get

$$\begin{aligned}\operatorname{osc}(u, S_\varphi(x_0, h/2)) &\le \operatorname{osc}(w, S_\varphi(x_0, h/2)) + \operatorname{osc}(v, S_\varphi(x_0, h/2))\\ &\le \beta \operatorname{osc}(u, S_\varphi(x_0, h)) + C_3 \|u\|_{L^\infty(S_\varphi(x_0,h_0))} h^{\gamma_2}\\ &\quad + C_1(\|\mathbf{F}\|_{L^\infty(S_\varphi(x_0,2h_0))} + \|f\|_{L^r(S_\varphi(x_0,2h_0))}) h^{\gamma_1}.\end{aligned}$$

Therefore, by a standard argument (see [20, Lemma 8.23]), for all $h \le h_0$ we get

$$\begin{aligned}&\operatorname{osc}(u, S_\varphi(x_0, h))\\ &\le C_4 \left(\frac{h}{h_0}\right)^{\gamma_3} \Big(\operatorname{osc}(u, S_\varphi(x_0, h_0)) + C_3 \|u\|_{L^\infty(S_\varphi(x_0,h_0))} h_0^{\gamma_2}\\ &\quad + C_1(\|\mathbf{F}\|_{L^\infty(S_\varphi(x_0,2h_0))} + \|f\|_{L^r(S_\varphi(x_0,2h_0))}) h_0^{\gamma_1}\Big)\\ &\le C_4 \left(\frac{h}{h_0}\right)^{\gamma_3} \Big((2 + C_3 h_0^{\gamma_2}) \|u\|_{L^\infty(S_\varphi(x_0,h_0))}\\ &\quad + C_1(\|\mathbf{F}\|_{L^\infty(S_\varphi(x_0,2h_0))} + \|f\|_{L^r(S_\varphi(x_0,2h_0))}) h_0^{\gamma_1}\Big),\end{aligned}$$

where $C_4 = C_4(\beta) > 0$ and $\gamma_3 = \gamma_3(\beta) > 0$. This gives the desired oscillation estimate (6.1). The proof of the Theorem is complete. □

Now, we combine the interior estimate in Lemma 5.2 with the Hölder estimate in Corollary 1.2 to prove Theorem 1.3.

Proof of Theorem 1.3 From Corollary 1.2, for all $x, y \in S_\varphi(x_0, h_0)$, we have

$$|u(x) - u(y)| \le C_1 \left(\|\mathbf{F}\|_{L^\infty(S_\varphi(x_0,2h_0))} + \|f\|_{L^r(S_\varphi(x_0,2h_0))} + \|u\|_{L^\infty(S_\varphi(x_0,h_0))}\right) |x - y|^\gamma, \tag{6.7}$$

where C_1 depends on $n, \lambda, \Lambda, r, \varepsilon^*, \|\mathbf{b}\|_{L^\infty(S_\varphi(x_0,2h_0))}, \|\mathbf{B}\|_{L^\infty(S_\varphi(x_0,2h_0))}, \|\operatorname{div} \mathbf{B}\|_{L^n(S_\varphi(x_0,2h_0))}$, h_0, and $\operatorname{diam}(S_\varphi(x_0, 4h_0))$, and γ depends on $n, \lambda, \Lambda, \varepsilon^*, \|\mathbf{b}\|_{L^\infty(S_\varphi(x_0,2h_0))}, \|\mathbf{B}\|_{L^\infty(S_\varphi(x_0,2h_0))}$, $\|\operatorname{div} \mathbf{B}\|_{L^n(S_\varphi(x_0,2h_0))}$, h_0, and $\operatorname{diam}(S_\varphi(x_0, 4h_0))$.

As

$$Du^+ = Du\,\chi_{\{u>0\}} \quad \text{and } Du^- = -Du\,\chi_{\{u<0\}},$$

u^+ and u^- are solutions to

$$\begin{aligned}-\operatorname{div}(\Phi Du^+ + u^+\mathbf{B}) + \mathbf{b} \cdot Du^+ &= f\chi_{\{u>0\}} - \operatorname{div}(\mathbf{F}\chi_{\{u>0\}}),\\ -\operatorname{div}(\Phi Du^- + u^-\mathbf{B}) + \mathbf{b} \cdot Du^- &= -f\chi_{\{u<0\}} + \operatorname{div}(\mathbf{F}\chi_{\{u<0\}}).\end{aligned}$$

Therefore, we may apply Lemma 5.2 to u^+ and u^- to get

$$\begin{aligned}\|u\|_{L^\infty(S_\varphi(x_0,h_0))} &\le C_2(h_0^{-\frac{n}{4}} \|u\|_{L^2(S_\varphi(x_0,2h_0))} + h_0^{1-\frac{n}{2}} \|\mathbf{F}\|_{L^\infty(S_\varphi(x_0,2h_0))}\\ &\quad + h_0^{1-\frac{n}{2r}} \|f\|_{L^r(S_\varphi(x_0,2h_0))}),\end{aligned} \tag{6.8}$$

where

$$C_2 = C_2(n, \lambda, \Lambda, r, \varepsilon^*, \|\mathbf{b}\|_{L^\infty(S_\varphi(x_0,2h_0))}, \|\mathbf{B}\|_{L^\infty(S_\varphi(x_0,2h_0))}, h_0, \operatorname{diam}(S_\varphi(x_0, 4h_0))) > 0.$$

Combining (6.7) and (6.8) completes the proof of the Theorem. □

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Declarations

Conflict of interest There is no Conflict of interest to declare.

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